THE KÄHLER-RICCI FLOW ON SURFACES OF POSITIVE KODAIRA DIMENSION

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1 Introduction

The existence of Kähler-Einstein metrics on a compact Kähler manifold has been the subject of intensive study over the last few decades, following Yau's solution to Calabi's conjecture (see [Ya2], [Au], [Ti1], [Ti2]). The Ricci flow, introduced by Richard Hamilton in [Ha1, Ha2], has become one of the most powerful tools in geometric analysis. The Ricci flow preserves the Kählerian property, so it provides a natural flow in Kähler geometry, referred as the Kähler-Ricci flow. Using the Kähler-Ricci flow, Cao [Ca] gave an alternative proof of the existence of Kähler-Einstein metrics on a compact Kähler manifold with trivial or negative first Chern class. In early 90's, Hamilton and Chow also used the Ricci flow to give another proof of

classical uniformization for Riemann surfaces (see [Ha2], [Ch]). Recently Perelman [Pe] has made a major breakthrough in studying the Ricci flow. The convergence of the Kähler-Ricci flow on Kähler-Einstein Fano manifolds was claimed by Perelman and it has been generalized to any Kähler manifolds admitting a Kähler-Ricci soliton by the second named author and Zhu [TiZhu]. Previously, in [ChTi], Chen and the second named author proved that the Kähler-Ricci flow converges to a Kähler-Einstein metric if the initial metric is of non-negative bisectional curvature.

However, most projective manifolds do not have a definite or trivial first Chern class. It is a natural question to ask if there exist any well-defined canonical metrics on these manifolds or on varieties canonically associated to them, i.e., canonical models. Tsuji [Ts] used the Kähler-Ricci flow to prove the existence of a canonical singular Kähler-Einstein metric on a minimal projective manifold of general type. In this paper, we propose a program of finding canonical metrics on canonical models of projective varieties of positive Kodaira dimension. We also carry out this program for minimal algebraic surfaces. To do it, we will study the Kähler-Ricci flow starting from any Kähler metrics and show its limiting behavior at time infinity.

Let X be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X. In local coordinates $z_1, ..., z_n$, we can write ω as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. Consider the Kähler-Ricci flow

$$\frac{\partial \omega(t,\cdot)}{\partial t} = -Ric(\omega(t,\cdot)) - \omega(t,\cdot), \qquad \omega(0,\cdot) = \omega_0, \tag{1.1}$$

where $\omega(t,\cdot)$ is a family of Kähler metrics on X and $Ric(\omega(t,\cdot))$ denotes the Ricci curvature of $\omega(t,\cdot)$ and ω_0 is a given Kähler metrics. If the canonical class K_X of X is ample and ω_0 represents K_X, Cao proved in [Ca] that (1.1) has a global solution $\omega(t,\cdot)$ for all $t > 0$ and $\omega(t,\cdot)$ converges to a Kähler-Einstein metric on X. If K_X is numerically effective (i.e. nef), Tsuji proved in [Ts] under additional assumption $[\omega_0] > K_X$ that (1.1) has a global solution $\omega(t,\cdot)$. This additional assumption was removed in [TiZha], moreover, if K_X is also big, $\omega(t,\cdot)$ converges to a singular Kähler-Einstein metric with locally bounded Kähler potentials as t tends to ∞ (see [TiZha]). Our problem is to show how $\omega(t,\cdot)$ behaves at time infinity.

If X is a minimal Kähler surface of non-negative Kodaira dimension, then K_X is numerically effective. The Kodaira dimension $\nu(X)$ of X is equal to $0, 1, 2$. If $\nu(X) = 0$, then a finite cover of X is either a K3 surface or a complex torus, then after an appropriate scaling, $\omega(t,\cdot)$ converges to the unique Calabi-Yau metric in the Kähler class $[\omega_0]$ (cf. [Ca]). If $\nu(X) = 2$, i.e., X is of general type, then $\omega(t,\cdot)$ converges to the unique Kähler-Einstein orbifold metric on its canonical model as t tends to ∞ (see [TiZha]). If $\nu(X) = 1$, then X is a minimal elliptic surface and does not admit any Kähler-Einstein current which is smooth outside a subvariety. Hence, one does not expect that $\omega(t,\cdot)$ converges to a smooth metric outside a subvariety of X.

In this paper, we study the limiting behavior of $\omega(t,\cdot)$ as t tends to ∞ in the case that X is a minimal elliptic surface. In its sequel, we will extend our results here to higher dimensional manifolds, that is, we will study the limiting behavior of (1.1) when X is a n-dimensional variety of Kodaira dimension in $(0, n)$ and with numerically positive K_X. Hence, our first question is to identify limiting candidates. Since X is a minimal elliptic surface, there is a holomorphic map $f : X \mapsto \Sigma$ such that $K_X = \pi^* L$ for some ample line bundle L over the curve Σ. Let Σ_{reg} consist of all $s \in \Sigma$ such that $f^{-1}(s)$ is smooth and let $X_{reg} = f^{-1}(\Sigma_{reg})$. For any $s \in \Sigma_{reg}$, $f^{-1}(s)$ is

an elliptic curve. Thus the L^2-metric on the moduli of elliptic curves induces a metric ω_{WP} on Σ_{reg}. We call a metric ω canonical if it is smooth on Σ_{reg} and extends appropriately to Σ and satisfies

$$Ric(\omega) = -\omega + \omega_{WP}, \quad \text{on } \Sigma_{reg}.$$

Such a metric exists and is unique in a suitable sense.[1] Here is our main result of this paper.

Theorem 1.1 *Let $f : X \to \Sigma$ be a minimal elliptic surface of $\nu(X) = 1$ with singular fibres $X_{s_1} = m_1 F_1, \ \dots \ , X_{s_k} = m_k F_k$ with multiplicity $m_i \in \mathbf{N}$, $i = 1, ..., k$. Then for any initial Kähler metric, the Kähler-Ricci flow (1.1) has a global solution $\omega(t, \cdot)$ for all time $t \in [0, \infty)$ satisfying:*

1. *$\omega(t, \cdot)$ converges to $f^*\omega_\infty \in -2\pi c_1(X)$ as currents for a positive current ω_∞ on Σ;*

2. *ω_∞ is smooth on Σ_{reg} and satisfies as currents on Σ*

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{WP} + \sum_{i=1}^{k} \frac{m_k - 1}{m_k}[s_i], \tag{1.2}$$

where ω_{WP} is the induced Weil-Petersson metric and $[s_i]$ is the current of integration associated to the divisor s_i on Σ, in particular, ω_∞ is a generalized Kähler-Einstein metric on Σ_{reg};

3. *for any compact subset $K \in X_{reg}$, there is a constant C_K such that*

$$||\omega(t, \cdot) - f^*\omega_\infty||_{L^\infty(K)} + e^t \sup_{s \in K} ||\omega(t, \cdot)|_{f^{-1}(s)}||_{L^\infty} \le C_K, \tag{1.3}$$

moreover, the scalar curvature of $\omega(t, \cdot)$ is uniformly bounded on any compact set of X_{reg}.

Remark 1.1 We conjecture that $\omega(t, \cdot)$ converges to $f^*\omega_\infty$ in the Gromov-Hausdorff topology and in the C^∞-topology outside singular fibers.

An elliptic surface $f : X \to \Sigma$ is an elliptic fibre bundle if it does not admit any singular fibre.

Corollary 1.1 *Let $f : X \to \Sigma$ be an elliptic fibre bundle over a curve Σ of genus greater one. Then the Kähler-Ricci flow (1.1) has a global solution with any initial Kähler metric. Furthermore, $\omega(t, \cdot)$ converges with uniformly bounded scalar curvature to the pullback of the Kähler-Einstein metric on Σ.*

This theorem seems to be the first general convergence result on collapsing of the Kähler-Ricci flow. Combining the results in [Ts, TiZha], we give a metric classification for Käher surfaces with an nef canonical line bundle by the Kähler-Ricci flow.

[1] Such canonical metrics can be also defined for higher dimensional manifolds. We refer the readers to section 5 for more details.

2 Preliminaries

2.1 Reduction of the Kähler-Ricci flow

In this section, we will reduce the Kähler-Ricci flow (1.1) to a parabolic equation on Kähler potentials for any compact Kähler manifold X with its canonical line bundle $K_X \geq 0$. Let X be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X. In local coordinates $z_1, ..., z_n$, ω can be written in the form of

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge dz_{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite hermitian matrix function. The curvature tensor for g is locally given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial z_{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial z_{\bar{l}}}, \qquad i, j, k = 1, 2, ..., n.$$

And the Ricci curvature is given by

$$R_{i\bar{j}} = -\frac{\partial^2 \log\det(g_{k\bar{l}})}{\partial z_i \partial z_{\bar{j}}}, \qquad i, j = 1, 2, ..., n.$$

So its Ricci curvature form is

$$Ric(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}} dz_i \wedge dz_{\bar{j}} = -\sqrt{-1}\partial\overline{\partial} \log\det(g_{k\bar{l}}).$$

Let $Ka(X)$ denote the Kähler cone of X, that is,

$$K_a(X) = \{[\omega] \in H^{1,1}(X, \mathbf{R}) \mid [\omega] > 0\}.$$

Suppose that $\omega(t, \cdot)$ is a solution of (1.1) on $[0, T)$. Then its induced equation on Kähler classes in $Ka(X)$ is given by the following ordinary differential equation

$$\begin{cases} \dfrac{\partial[\omega]}{\partial t} = -2\pi c_1(X) - [\omega] \\ [\omega]|_{t=0} = [\omega_0]. \end{cases} \tag{2.1}$$

It follows

$$[\omega(t, \cdot)] = -2\pi c_1(X) + e^{-t}([\omega_0] + 2\pi c_1(X)).$$

Now if we assume that X has semi-positive canonical bundle K_X, then there is a large integer μ such that any basis of $H^0(X, \mu K_X)$ gives rise to an embedding f into a projective space. Recall the Kodaira dimension $\nu(X)$ of X is defined to be the dimension of the image of this embedding. This dimension is in fact independent of choices of the basis and embedding. Moreover, using this embedding, one can see easily that there is a positive $(1,1)$-form χ such that $f^*\chi$ represents $-2\pi c_1(X)$. Choose the reference Kähler metric ω_t by

$$\omega_t = \chi + e^{-t}(\omega_0 - \chi). \tag{2.2}$$

Here we abuse the notation by identifying χ and $f^*\chi$ for simplicity. Then the solution of (1.1) can be written as

$$\omega = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi.$$

We can always choose a smooth volume form Ω on X such that $Ric(\Omega) = \chi$. Then the evolution for the Kähler potential φ is given by the following initial value problem:

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-\nu(X))t}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^2}{\Omega} - \varphi \\ \varphi|_{t=0} = 0. \end{cases} \tag{2.3}$$

The following was proved in [TiZha]. When $\omega_0 > \chi$ and $K_X \geq 0$, it was proved by Tsuji in [Ts]. It was also proved in [CaLa] under a stronger assumption.

Theorem 2.1 *The Kähler-Ricci flow (2.3) has a global solution for all time $t \in [0, \infty)$ if K_X is nef.*

The evolution equation for the scalar curvature R is given by

$$\frac{\partial R}{\partial t} = \Delta R + |Ric|^2 + R. \tag{2.4}$$

Then the following proposition is an immediate conclusion from the maximum principle for the parabolic equation (2.4).

Proposition 2.1 *The scalar curvature along the Kähler-Ricci flow (1.1) is uniformly bounded from below.*

2.2 Minimal surfaces with positive Kodaira dimension

An elliptic fibration of a surface X is a proper, connected holomorphic map $f : X \to \Sigma$ from X to a curve Σ such that the general fibre is a non-singular elliptic curve. An elliptic surface is a surface admitting an elliptic fibration. Any surface X of $\nu(X) = 1$ must be an elliptic surface. Such an elliptic surfaces is sometimes called a properly elliptic surface. Since we assume that X is minimal, all fibres are free of (-1)-curves. A very simple example is the product of two curves, one elliptic and the other of genus ≥ 2.

Let $f : X \to \Sigma$ be an elliptic surface. The differential *df* can be viewed as an injection of sheaves $f^*(K_\Sigma) \to \Omega^1_X$. Its cokernel $\Omega_{X/\Sigma}$ is called the sheaf of relative differentials. In general, $\Omega_{X/\Sigma}$ is far from being locally free. If some fibre has a multiple component, then *df* vanishes along this component and $\Omega_{X/\Sigma}$ contains a torsion subsheaf with one-dimensional support. Away from the singularities of f we have the following exact sequence

$$0 \to f^*(K_\Sigma) \to \Omega^1_X \to \Omega_{X/\Sigma} \to 0$$

including an isomorphism between $\Omega_{X/\Sigma}$ and $K_X \otimes f^*(K_\Sigma^\vee)$. We also call the line bundle $\Omega_{X/\Sigma}$ the dualizing sheaf of f on X. The following theorem is well-known (cf. [BaHuPeVa]).

Theorem 2.2 *Let $f : X \to \Sigma$ be a minimal elliptic surface such that its multiple fibres are $X_{s_1} = m_1 F_1$, .., $X_{s_k} = m_k F_k$. Then*

$$K_X = f^*(K_\Sigma \otimes (f_{*1}\mathcal{O}_X)^\vee) \otimes \mathcal{O}_X(\sum (m_i - 1)F_i), \tag{2.5}$$

or

$$K_X = f^*(L \otimes \mathcal{O}_X(\sum (m_i - 1)F_i), \tag{2.6}$$

where L is a line bundle of degree $\chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_\Sigma)$ on Σ.

Note that $\deg(f_{*1}\mathcal{O}_X)^\vee = \deg(f_*\Omega_{X/\Sigma}) \geq 0$ and the equality holds if and only if f is locally trivial. The following invariant

$$\delta(f) = \chi(\mathcal{O}_X) + \left(2g(\Sigma) - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i})\right)$$

determines the Kodaira dimension of X.

Proposition 2.2 (cf. [BaHuPeVa]) *Let $f : X \to \Sigma$ be a relatively minimal elliptic fibration and X be compact. Then $\nu(X) = 1$ if and only if $\delta(f) > 0$.*

Kodaira classified all possible singular fibres for f. A fibre X_s is stable if

1. X_s is reduced,
2. X_s contains no (-1)-curves,
3. X_s has only node singularities.

The only stable singular fibres are of type I_b for $b > 0$, therefore such singular fibres are particularly interesting. Let $\mathcal{S}_1/\Gamma_1 \cong \mathbf{C}$ be the period domain, where $\mathcal{S}_1 = \{z \in \mathbf{C} \mid \mathrm{Im} z > 0\}$ is the upper half plane and $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})/\{\pm 1\}$ is the modular group acting by $z \to \frac{az+b}{cz+d}$. The j-function gives an isomorphism $\mathcal{S}_1/\Gamma_1 \to \mathbf{C}$ with

1. $j(z) = 0$ if $z = e^{\frac{\pi}{3}\sqrt{-1}}$ modular Γ_1,
2. $j(z) = 1$ if $z = \sqrt{-1}$ modular Γ_1.

Thus any elliptic surface $f : X \to \Sigma$ gives a period map $p : \Sigma_{reg} \to \mathcal{S}_1/\Gamma_1$. Set $J : \Sigma_{reg} \mapsto \mathbf{C}$ by $J(s) = j(p(s))$. For a stable fibre X_s of type I_b, the functional invariant J has a pole of order b at s and the monodromy is given by $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Now choose a semi-positive $(1,1)$-form $\chi \in -2\pi c_1(X)$ to be the pullback of a Kähler form χ on the base Σ and $\chi = f^*\chi$ might vanish somewhere due to the presence of singular fibres. Then as we discussed in the previous subsection, one can reduce the Kähler-Ricci flow (1.1) to an evolution equation on Kähler potentials. Hence, the Kähler-Ricci flow (1.1) has a global solution and provides a canonical way of deforming any given Kähler metric to a canonical metric on the canonical model of minimal elliptic surfaces of positive Kodaira dimension. As described in Theorem 1.1, this canonical metric on Σ satisfies the curvature equation

$$Ric(g_\infty) = -g_\infty + g_{WP} + \sum_{i=1}^{k} \frac{m_i - 1}{m_i}[s_i].$$

It corresponds to (2.6), where the pullback of the Weil-Petersson metric g_{WP} by the period map p is the curvature of the dualizing sheaf $f_*\Omega_{X/\Sigma}$ and the current $\sum_{i=1}^{k} \frac{m_i-1}{m_i}[s_i]$ corresponds to the residues from the multiple fibres.

3 A parabolic Schwarz lemma

In this section we will establish a parabolic Schwarz lemma for compact Kähler manifolds. This is inspired by [Ya1]. This will lead us to identify and estimate the collapsing on the vertical direction for properly minimal elliptic surfaces and in general certain fibre spaces. It also plays a key role in estimating the scalar curvature along the Kähler-Ricci flow. Let $f : X \to Y$ be a non-constant holomorphic mapping between two Kähler manifolds. Suppose $\dim X = n$ and the Kähler metric $\omega(t, \cdot)$ on X is deformed by the following Kähler-Ricci flow (1.1). Then we have the following parabolic Schwarz lemma for metrics.

Theorem 3.1 *If the holomorphic bisectional curvature of Y with respect to a fixed Kähler metric $h_{\alpha\overline{\beta}}$ is bounded from above by a negative constant $-K$ and the Kähler-Ricci flow (1.1) exists for all $t \in [0, T)$, then*

$$f^*h \leq \frac{C(t)}{K}\omega(t, \cdot) \ , \tag{3.1}$$

where $C(t)$ is a positive function in t dependent on the initial metric ω_0 and $\lim_{t\to\infty} C(t) = 1$ if $T = \infty$.

Proof Choose normal coordinate systems for $g = \omega(t, \cdot)$ on X and h on Y respectively. Let $u = tr_g(h) = g^{i\overline{j}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}$ and we will calculate the evolution of u.

$$\begin{aligned}
\Delta u &= g^{k\bar{l}}\partial_k\partial_{\bar{l}}(g^{i\overline{j}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}}) \\
&= g^{i\bar{l}} g^{k\overline{j}} R_{k\bar{l}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} + g^{i\overline{j}} g^{k\bar{l}} f_{i,k}^\alpha f_{\overline{j},\bar{l}}^{\overline{\beta}} h_{\alpha\overline{\beta}} - g^{i\overline{j}} g^{k\bar{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} f_k^\gamma f_{\bar{l}}^{\overline{\delta}},
\end{aligned}$$

where $S_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is the curvature tensor of $h_{\alpha\bar{\beta}}$ and the Laplacian Δ acts on functions ϕ by

$$\Delta\phi = g^{i\overline{j}}\partial_i\partial_{\overline{j}}\phi.$$

By the definition of u we have

$$\Delta u \geq g^{i\bar{l}} g^{k\overline{j}} R_{k\bar{l}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} + Ku^2.$$

Now

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -g^{i\bar{l}} g^{k\overline{j}} \frac{\partial g_{k\bar{l}}}{\partial t} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} \\
&= g^{i\bar{l}} g^{k\overline{j}} (R_{k\bar{l}} + g_{k\bar{l}}) f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} \\
&= g^{i\bar{l}} g^{k\overline{j}} R_{k\bar{l}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} h_{\alpha\overline{\beta}} + u,
\end{aligned}$$

therefore

$$(\frac{d}{dt} - \Delta)u \leq u - Ku^2. \tag{3.2}$$

Applying the maximal principle, we have

$$\frac{d}{dt}u_{max} \leq u_{max} - Ku_{max}^2.$$

Thus

$$u_{max}(t) \leq \frac{1}{K - Ce^{-t}}$$

and it proves the theorem. □

By similar argument as in the proof of Theorem 3.1 one can also derive the following Schwarz lemma for volume forms with weaker curvature bounds on the target manifold.

Theorem 3.2 *Suppose* $\dim X = n \geq \dim Y = \kappa$. *Let* χ *be the Kähler form on* Y *with respect to the Kähler metric* $h_{\alpha\overline{\beta}}$. *If* $Ric(h) \leq -Kh$ *for some* $K > 0$ *and the Kähler-Ricci flow (1.1) exists for all* $t \in [0, T)$, *then there exists a constant* $C > 0$ *dependent on the initial metric* ω_0 *such that*

$$\frac{\omega^{n-\kappa} \wedge f^*\chi^\kappa}{\omega^n} \leq C. \tag{3.3}$$

Suppose $2\pi c_1(X) = -[f^*\chi]$ for a Kähler form χ on Y, i.e. K_X is a semi-positive line bundle pulled back from an ample line bundle from Y. Then we can remove the curvature assumption on Y. From now on, for convenience, we will write $f^*\chi$ as χ. Since $c_1(X) \leq 0$, by Theorem 2.1, the Käher-Ricci flow has long time existence.

Theorem 3.3 *Let* $f : X \to Y$ *be a holomorphic fibration such that* $2\pi c_1(X) = -[f^*\chi]$ *for some Kähler form* χ *on* Y. *Then the Kähler-Ricci flow (1.1) exists for all* $t \in [0, \infty)$ *and there exist constants* $A, C > 0$ *such that for all* (t, z),

$$f^*\chi(t,z) \leq C\omega(t,z)e^{A\varphi(t,z)} \max_{[0,t]\times X} \{\log \frac{\Omega}{e^{(n-\kappa)s}\omega(s,\cdot)^n} e^{-A\varphi}\}, \tag{3.4}$$

where Ω *is the volume form on* X *such that* $Ric(\Omega) = f^*\chi$.

Proof Let $u = g^{i\overline{j}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}}$ and choose normal coordinates for g and χ. We will calculate the evolution for $\log u$. Note that $\Delta \log u = \frac{\Delta u}{u} - \frac{|\nabla u|_g^2}{u^2}$ and

$$\Delta u = g^{i\overline{l}} g^{k\overline{j}} R_{k\overline{l}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} + g^{k\overline{l}} g^{i\overline{j}} f_{i,k}^\alpha f_{\overline{j},\overline{l}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}} - g^{i\overline{j}} g^{k\overline{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} f_k^\gamma f_{\overline{l}}^{\overline{\delta}}. \tag{3.5}$$

Applying the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\nabla u|_g^2 &= \sum_{i,j,k,\alpha,\beta} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} f_{i,k}^\alpha f_{\overline{j},\overline{k}}^{\overline{\beta}} \\
&\leq \sum_{i,j,\alpha,\beta} |f_i^\alpha f_{\overline{j}}^{\overline{\beta}}| (\sum_k |f_{i,k}^\alpha|^2)^{\frac{1}{2}} (\sum_l |f_{\overline{j},\overline{l}}^{\overline{\beta}}|^2)^{\frac{1}{2}} \\
&= (\sum_{i,\alpha} |f_i^\alpha| (\sum_k |f_{i,k}^\alpha|^2)^{\frac{1}{2}})^2 \\
&\leq (\sum_{j,\beta} |f_j^\beta|^2)(\sum_{i,k,\alpha} |f_{i,k}^\alpha|^2).
\end{aligned}$$

Let C be a constant satisfing $S_{\alpha\overline{\beta}\gamma\overline{\delta}} \leq C\chi_{\alpha\overline{\beta}}\chi_{\gamma\overline{\delta}}$. Then we have

$$\begin{aligned}
& (\frac{d}{dt} - \Delta) \log u \\
= \; & \frac{1}{u}(-g^{k\overline{l}} g^{i\overline{j}} f_{i,k}^\alpha f_{\overline{j},\overline{l}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}} + g^{i\overline{j}} g^{k\overline{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} f_k^\gamma f_{\overline{l}}^{\overline{\delta}} + \frac{|\nabla u|_g^2}{u}) + 1 \\
\leq \; & -\frac{1}{u} g^{i\overline{j}} g^{k\overline{l}} S_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} f_k^\gamma f_{\overline{l}}^{\overline{\delta}} + 1 \\
\leq \; & -Cu + 1.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(\frac{d}{dt}-\Delta)\varphi &= -tr_\omega(\sqrt{-1}\partial\overline{\partial}\varphi)+\frac{\partial\varphi}{\partial t}\\
&= -tr_\omega(\omega-\omega_t)+\frac{\partial\varphi}{\partial t}\\
&= tr_\omega(\omega_t)+\frac{\partial\varphi}{\partial t}-n.
\end{aligned}
$$

Notice that φ is uniformly bounded from above from the equation (2.3) by the maximum principle. Combining the above estimates we have

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)(\log u-2A\varphi)\\
\le\ & -2(A-C)u-2A\log\frac{e^{(n-\kappa)t}\omega^n}{\Omega}+2A\varphi+2nA+1\\
\le\ & -Au+2A\log\frac{\Omega}{e^{(n-\kappa)t}\omega^n}+2A\varphi+2nA+1\\
\le\ & -Au+2A\log\frac{\Omega}{e^{(n-\kappa)t}\omega_t^n}+CA
\end{aligned}
$$

if we choose A sufficiently large.

Suppose on each time interval $[0,t]$, the maximum of $\log u-A\varphi$ is achieved at (t_0,z_0), by the maximum principle we have

$$u(t_0,z_0)\le 2\log\frac{\Omega}{e^{(n-\kappa)t_0}\omega^n}(t_0,z_0)$$

and

$$u(t,z)\le u(t_0,z_0)e^{2A\varphi(z,t)-2A\varphi(t_0,z_0)}.$$

This completes the proof. □

4 Estimates

In this section, we prove the uniform zeroth order and second order estimate of the potential φ along the Kähler-Ricci flow. A gradient estimate is also derived and it gives a uniform bound of the scalar curvature. We assume that $f: X\to\Sigma$ is a properly minimal surface over a curve Σ with singular fibres over $\Delta=\{s_1,...,s_k\}\subset\Sigma$. Let $X_{s_i}=f^{-1}(s_i)$ be the corresponding singular fibre for $i=1,...,k$ and $[S]=\sum_{i=1}^k[X_{s_i}]=f^*(\sum_{i=1}^k[s_i])$ be the divisor containing all the singular fibres. We can always find a hermitian metric h on the line bundle induced by $[S]$ such that $Ric(h)$ is a multiple of χ.

4.1 The zeroth order and volume estimates

We will derive the zeroth order estimates for φ and $\frac{d\varphi}{dt}$.

Lemma 4.1 *Let φ be a solution of the Kähler-Ricci flow (2.3). There exists a positive constant C depending only on the initial data such that $\varphi\le C$.*

Proof This is a straightforward application of the maximum principle. Let $\varphi_{max}(t) = \max_X \varphi(t, \cdot)$. Applying the maximum principle, we have

$$\begin{aligned}
\frac{\partial \varphi_{max}}{\partial t} &\leq \log \frac{e^t \omega_t^2}{\Omega} - \varphi_{max} \\
&\leq \log \frac{2\chi \wedge (\omega_0 - \chi) + e^{-t}(\omega_0 - \chi)^2}{\Omega} - \varphi_{max} \\
&\leq C - \varphi_{max}.
\end{aligned}$$

This gives a uniform upper bound for φ. □

Lemma 4.2 *There exists a positive constant C depending only on the initial data such that*

$$\frac{\partial \varphi}{\partial t} \leq C. \tag{4.1}$$

Proof Differentiating on both sides of (2.3) we obtain

$$\frac{\partial}{\partial t}(\frac{\partial \varphi}{\partial t}) = \Delta \frac{\partial \varphi}{\partial t} + 1 - e^{-t} tr_\omega(\omega_0 - \chi) - \frac{\partial \varphi}{\partial t}, \tag{4.2}$$

where Δ is the Laplacian operator of the metric g. It can be rewritten as

$$\frac{\partial}{\partial t}(e^t \frac{\partial \varphi}{\partial t}) = \Delta(e^t \frac{\partial \varphi}{\partial t}) + e^t - tr_\omega(\omega_0 - \chi),$$

and

$$\frac{\partial}{\partial t}(\frac{\partial \varphi}{\partial t} + \varphi) = \Delta(\frac{\partial \varphi}{\partial t} + \varphi) + tr_\omega(\chi) - 1. \tag{4.3}$$

So

$$\frac{\partial}{\partial t}(e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t) = \Delta(e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t) - tr_\omega(\omega_0).$$

Applying the maximum principle, we have

$$e^t \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} - \varphi - e^t - t \leq C'$$

for some uniform constant C' only depending on the initial data. Hence

$$\frac{\partial \varphi}{\partial t} \leq \frac{e^{-t}}{1 - e^{-t}} \varphi + C' \leq C.$$

□

Lemma 4.3 *There exists a positive constant C depending only on the initial data such that*

$$|\varphi| \leq C. \tag{4.4}$$

Proof It suffices to derive the lower bound for φ. Consider $u(t, z) = \max_X \varphi(t, \cdot) - \varphi(t, z) \geq 0$. Fix $\delta > 0$. For any $p > 1$, since both φ and $\frac{\partial \varphi}{\partial t}$ are bounded from above, using equation (2.3), we have

$$\int_X e^{p\delta u}(\omega^2 - \omega_t^2) \leq \int_X e^{p\delta u} \omega^2 \leq C e^{-t} \int_X e^{p\delta u} \omega_0^2. \tag{4.5}$$

Calculate

$$
\begin{aligned}
& \int_X e^{p\delta u}(\omega^2 - \omega_t^2) \\
= \;& \sqrt{-1}\int_X e^{p\delta u}\partial\overline{\partial}(-u)\wedge(\omega+\omega_t) \\
= \;& \frac{2\sqrt{-1}}{p\delta}\int_X \partial e^{\frac{p}{2}\delta u}\wedge\overline{\partial}e^{\frac{p}{2}\delta u}\wedge(\omega+\omega_t) \\
\geq \;& \frac{2\sqrt{-1}}{p\delta}\int_X \partial e^{\frac{p}{2}\delta u}\wedge\overline{\partial}e^{\frac{p}{2}\delta u}\wedge\omega_t \\
\geq \;& \frac{\sqrt{-1}C}{p\delta}e^{-t}\int_X \partial e^{\frac{p}{2}\delta u}\wedge\overline{\partial}e^{\frac{p}{2}\delta u}\wedge\omega_0.
\end{aligned} \tag{4.6}
$$

Combining (4.5) and (4.6) we obtain

$$\int_X |\nabla e^{\frac{p}{2}\delta u}|^2\omega_0^2 \leq Cp\int_X e^{p\delta u}\omega_0^2.$$

The Sobolev inequality $||f||_{L^4}^2 \leq C||f||_{H^1}^2$ implies that for all $p > 1$

$$||e^{\delta u}||_{L^{2p}}^p \leq C\delta p||e^{\delta u}||_{L^p}^p.$$

Now we can apply Moser's iteration by successively replacing p by 2^k and let $k \to \infty$. Then the standard argument shows that

$$||e^{\delta u}||_{L^\infty} \leq C||e^{\delta u}||_{L^1}.$$

Then we only need to bound the quantity $||e^{\delta u}||_{L^1}$. Note that $A\omega_0 - \sqrt{-1}\partial\overline{\partial}u \geq \chi + e^{-t}(\omega_0 - \chi) + \sqrt{-1}\partial\overline{\partial}\varphi > 0$ if we choose $A > 0$ sufficiently large. The lemma is proved if we apply the following proposition. It is proved by the second named named author in [Ti1] based on a result in [Hö].

Proposition 4.1 *There exists $\delta > 0$ and C depending only on (X, ω_0) such that*

$$\int_X e^{-\delta\phi}\omega_0^n \leq C, \tag{4.7}$$

for all $\phi \in C^2(X)$ satisfying $\omega_0 + \sqrt{-1}\partial\overline{\partial}\phi > 0$ and $\sup_X \phi = 0$.

This completes the proof. □

Since $e^t\omega^2 = e^{\frac{\partial\varphi}{\partial t}+\varphi}\Omega$ and $||\varphi||_{C^0}$ is uniformly bounded, from the uniform upper bound for $\frac{\partial\varphi}{\partial t}$ we conclude that the normalized volume form $e^t\omega^2$ is uniformly bounded above and a lower bound for it will also give a lower bound for $\frac{\partial\varphi}{\partial t}$.

Lemma 4.4 *There exist constants $\lambda_1 > 0$ and $C > 0$ such that for all $(t, z) \in [0, \infty) \times X$ we have the following volume estimate*

$$\frac{1}{C}|S|_h^{2\lambda_1} \leq \frac{e^t\omega^2}{\Omega} \leq C.$$

Here h is a fixed hermitian metric equipped on the line bundle induced by the divisor $[S]$ such that $Ric(h) > 0$ is a multiple of χ.

Proof It suffices to prove the lower bound of the volume form $e^t\omega^2$. Notice that $\log \frac{e^t\omega^2}{\Omega} = \frac{\partial\varphi}{\partial t}+\varphi$ and hence the evolutions for $\log \frac{e^t\omega^2}{\Omega}$ and φ are prescribed by

$$(\frac{\partial}{\partial t} - \Delta) \log \frac{e^t\omega^2}{\Omega} = tr_\omega(\chi) - 1 \ \ \text{and} \tag{4.8}$$

$$(\frac{\partial}{\partial t} - \Delta)\varphi = tr_\omega(\omega_t) + \log \frac{e^t\omega^2}{\Omega} - \varphi - 2. \tag{4.9}$$

Combining the above equations, at any point (t, z) we have for $\lambda > 0$

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)(\log \frac{e^t\omega^2}{\Omega} + 2A\varphi - \lambda \log |S|_h^2) \\
= \ & 2Atr_\omega(\omega_t) + tr_\omega(\chi) - \lambda tr_\omega(Ric(h)) + 2A \log \frac{e^t\omega^2}{\Omega} - 2A\varphi - (4A + 1) \\
\geq \ & Atr_\omega(\omega_t) + 2A \log \frac{e^t\omega^2}{\Omega} + tr_\omega(A\omega_t - \lambda Ric(h)) - C(A + 1) \\
\geq \ & Atr_\omega(\omega_t) + 2A \log \frac{e^t\omega^2}{\Omega} - C(A + 1)
\end{aligned}$$

if we choose A sufficiently large. Suppose on each time interval $[0, T]$, the minimum of $\log \frac{e^t\omega^2}{\Omega} + 2A\varphi - \lambda \log |S|_h^2$ is achieved at (t_0, z_0), then by the maximum principle at (t_0, z_0) we have

$$tr_\omega(\omega_t)(t_0, z_0) \leq 2 \log \frac{\Omega}{e^t\omega^2}(t_0, z_0) + C. \tag{4.10}$$

But for some $\lambda > 0$ we have at (t_0, z_0)

$$C + 2 \log \frac{\Omega}{e^t\omega^2} \geq tr_\omega(\omega_t) \geq (\frac{\omega_t^2}{\omega^2})^{\frac{1}{2}} \geq (\frac{\Omega}{e^t\omega^2})^{\frac{1}{2}} (\frac{\chi \wedge \omega_0}{\Omega})^{\frac{1}{2}} \geq C(|S|_h^{2\lambda} \frac{\Omega}{e^t\omega^2})^{\frac{1}{2}}.$$

For each $\delta > 0$, there is the following elementary inequality

$$\log x \leq x^\delta + C_\delta \quad \text{ for all } x > 0.$$

It follows that at (t_0, z_0), we have for some small $\delta < \frac{1}{2}$

$$(|S|_h^{2\lambda} \frac{\Omega}{e^t\omega^2})^{\frac{1}{2}} \leq C((\frac{\Omega}{e^t\omega^2})^\delta + 1)$$

and by multiplying $|S|_h^{2\delta\lambda_1}$,

$$(|S|_h^{2\lambda+4\delta\lambda_1} \frac{\Omega}{e^t\omega^2})^{\frac{1}{2}} \leq C((|S|_h^{2\lambda_1} \frac{\Omega}{e^t\omega^2})^\delta + 1).$$

We have $2\lambda + 4\delta\lambda_1 = 2\lambda_1$ if λ_1 is chosen by $\lambda_1 = \frac{\lambda}{1-2\delta}$. Therefore $|S|_h^{2\lambda_1} \frac{\Omega}{e^{t_0}\omega^2}(t_0, z_0) \leq C$ and

$$\frac{e^t\omega^2}{|S|_h^{2\lambda_1}\Omega} e^\varphi(t, z) \geq \frac{e^{t_0}\omega^2}{|S|_h^{2\lambda_1}\Omega} e^\varphi(t_0, z_0).$$

Both φ and $\frac{e^{t_0}\omega^2}{|S|_h^{2\lambda_1}\Omega}(t_0, z_0)$ are uniformly bounded from below, hence the lemma is proved. $\square$

This also shows that there is a uniform lower bound for $\frac{\partial\varphi}{\partial t}$ with at worse log poles near the singular fibres.

Lemma 4.5 *There exists a constant $C > 0$ such that*

$$\frac{\partial\varphi}{\partial t} \geq C(\lambda_1 \log |S|_h^2 - 1). \tag{4.11}$$

Proof We only have to show $\frac{\partial\varphi}{\partial t}$ is uniformly bounded from below. This is obtained by the previous lemma and

$$\frac{\partial\varphi}{\partial t} = \log \frac{e^t\omega^2}{\Omega} - \varphi.$$

□

4.2 A partial second order estimate

In this section, we slightly modify the proof of the parabolic Schwarz lemma to derive a partial second order estimate. This will imply that along the Kähler-Ricci flow (1.1) the metric collapses along the fiber direction exponentially fast outside the singular fibres.

Lemma 4.6 (The partial second order estimate) *For any $\delta > 0$ there exists a constant $C > 0$ depending on δ such that*

$$tr_\omega(\chi) \leq \frac{C}{|S|_h^{2\delta}}. \tag{4.12}$$

Proof By Lemma 4.5, for any $\delta > 0$

$$|S|^{2\delta}\frac{\partial\varphi}{\partial t} \leq C.$$

Let $u = g^{i\overline{j}} f_i^\alpha f_{\overline{j}}^{\overline{\beta}} \chi_{\alpha\overline{\beta}}$. Following the similar calculation in Section 3, we have

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)(\log |S|^{2\delta} u - 3A\varphi) \\
\leq\ & -2Au - 3A\frac{\partial\varphi}{\partial t} + \delta tr_\omega(Ric(h)) + CA \\
\leq\ & -Au - 3A\frac{\partial\varphi}{\partial t} + CA
\end{aligned}$$

for sufficiently large A. Suppose on each time interval $[0, T]$, the maximum of $\log |S|^{2\delta} u - A\varphi$ is achieved at (t_0, z_0), by the maximum principle we have

$$(|S|^{2\delta} u)(t_0, z_0) \leq -3(|S|^{2\delta}\frac{\partial\varphi}{\partial t})(t_0, z_0) + C \leq C.$$

Combining with the uniform bound of $|\varphi|$, we can conclude that $|S|^{2\delta} u$ is uniformly bounded and the theorem is proved. □

Corollary 4.1 *Let X_s be a non-singular fibre for any $s \in \Sigma_{reg}$. Then along the Kähler-Ricci flow (1.1), ω decays exponentially fast on X_s. Furthermore if Δ_s is the Laplacian on X_s with respect to $\omega_0|_{X_s}$, then there exist constants $\lambda_2 > 0$ and $C > 0$ such that*

$$-e^{-t} \leq \Delta_s\varphi \leq \frac{Ce^{-t}}{|S|^{2\lambda_2}(s)}. \tag{4.13}$$

Proof Applying the partial second order estimate, we have

$$0 < e^{-t} + \Delta_s\varphi = \frac{\omega|_{X_s}}{\omega_0|_{X_s}} = \frac{\omega\wedge\chi}{\omega_0\wedge\chi} = \frac{\omega\wedge\chi}{\omega^2}\frac{\omega^2}{\omega_0\wedge\chi} \le tr_\omega(\chi)\frac{\omega^2}{\omega_0\wedge\chi} \le \frac{Ce^{-t}}{|S|^{2\lambda_2}(s)}$$

for some uniform constants C and λ_2. This proves the corollary. □

The partial second-order estimate enables us to derive the following strong partial C^0-estimate.

Corollary 4.2 *There exists constants $\lambda_3 > 0$ and $C > 0$ such that for all $s \in \Sigma_{reg}$*

$$|\sup_{X_s}\varphi - \inf_{X_s}\varphi| \le \frac{Ce^{-t}}{|S|_h^{2\lambda_3}(s)}.$$

Proof Let $\theta(s)$ be the smooth family of standard flat metrics on the elliptic fibres over Σ_{reg} such that $\int_{X_s}\theta(s) = \int_{X_s}\omega_0$ for all $s\in\Sigma_{reg}$. Let $\Delta_{\theta(s)}$ be the Laplacian of $\theta(s)$ on each smooth fibre X_s. By Green's formula, we have

$$\varphi - \frac{1}{\int_{X_s}\theta(s)}\int_{X_s}\varphi\theta(s) = \int_{X_s}\Delta_{\theta(s)}\varphi(y)(G_s(x,y)+A_s)\theta(s),$$

where $G_s(\cdot,\cdot)$ is Green's function with respect to $\theta(s)$ and $A_s = \inf_{X_s\times X_s} G_s(\cdot,\cdot)$. Since $(X_s,\theta(s))$ is a flat torus, one can easily show that Green's function $G_s(\cdot,\cdot)$ is uniformly bounded below by a multiple of $\mathrm{Diam}^2(X_s,\theta(s))$. However the diameter $\mathrm{diam}(X_s,\theta(s))$ might blow up near the singular fibres and actually there exists $\lambda > 0$ such that

$$\mathrm{diam}(X_s,\theta(s)) \le \frac{C}{|S(s)|_h^\lambda}.$$

Therefore $A_s \ge -\frac{C}{|S(s)|_h^{2\lambda}}$ for some constant C and we have on each smooth fibre X_s,

$$|\sup_{X_s}\varphi - \inf_{X_s}\varphi| \le C\sup_{X_s}|\Delta_{\theta(s)}\varphi||S(s)|_h^{-2\lambda}.$$

But for some $\mu > 0$ and $C > 0$ we have

$$|\Delta_{\theta(s)}\varphi| = |\Delta_s\varphi\frac{\omega_0|_{X_s}}{\theta(s)}| = |\Delta_s\varphi|\left.\frac{\omega_0\wedge\chi}{\theta(s)\wedge\chi}\right|_{X_s} \le \frac{Ce^{-t}}{|S(s)|_h^\mu},$$

where the last inequality follows from Corollary 4.1 and Lemma 5.4. This completes the proof of the corollary. □

4.3 Gradient estimates

In this section we will adapt the arguments in [ChYa] to obtain a uniform bound for $\left|\nabla\frac{\partial\varphi}{\partial t}\right|_g$ and the scalar curvature R. Let $u = \frac{\partial\varphi}{\partial t} + \varphi = \log\frac{e^t\omega^2}{\Omega}$. The evolution equation for u is given by

$$\frac{\partial u}{\partial t} = \Delta u + tr_\omega(\chi). \tag{4.14}$$

We will obtain a gradient estimate for u, which will help us bound the scalar curvature from below. Note that u is uniformly bounded from above, so we can find a constant $A > 0$ such that $A - u \ge 1$.

Theorem 4.1 *There exist positive integers λ_4, λ_5 and a uniform constant $C > 0$ such that*

(i) $|S|_h^{2\lambda_4} |\nabla u|^2 \leq C(A-u)$,

(ii) $-|S|_h^{2\lambda_5} \Delta u \leq C(A-u)$,

where ∇ is the gradient operator with respect to the metric g.

Proof Standard computation gives the following evolution equations for $|\nabla u|^2$ and Δu:

$$(\frac{\partial}{\partial t} - \Delta)|\nabla u|^2 = |\nabla u|^2 + (\nabla tr_\omega(\chi) \cdot \overline{\nabla} u + \overline{\nabla} tr_\omega(\chi) \cdot \nabla u) - |\nabla\nabla u|^2 - |\overline{\nabla}\nabla u|^2 \tag{4.15}$$

$$(\frac{\partial}{\partial t} - \Delta)\Delta u = \Delta u + g^{i\bar{l}} g^{k\bar{j}} R_{k\bar{l}} u_{i\bar{j}} + \Delta tr_\omega(\chi). \tag{4.16}$$

On the other hand, $\nabla_i \overline{\nabla}_{\bar{j}} u = -R_{i\bar{j}} - \chi_{i\bar{j}}$, hence

$$(\frac{\partial}{\partial t} - \Delta)\Delta u = \Delta u - |\nabla\overline{\nabla} u|^2 - g^{i\bar{l}} g^{k\bar{j}} \chi_{i\bar{j}} u_{k\bar{l}} + \Delta tr_\omega(\chi).$$

We shall now prove the first inequality. Let

$$H = |S|_h^{2\lambda_4} (\frac{|\nabla u|^2}{A-u} + tr_\omega(\chi)).$$

The evolution equation for H is given by

$$\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)(|S|_h^{2\lambda_4} (\frac{|\nabla u|^2}{A-u})\\
= \;& |S|_h^{2\lambda_4} \frac{|\nabla u|^2 - |\nabla\nabla u|^2 - |\overline{\nabla}\nabla u|^2 + (\nabla tr_\omega(\chi) \cdot \overline{\nabla} u + \overline{\nabla} tr_\omega(\chi) \cdot \nabla u)}{A-u}\\
& -\Delta|S|_h^{2\lambda_4} \frac{|\nabla u|^2}{A-u} - [\nabla|S|_h^{2\lambda_4} (\frac{\overline{\nabla}|\nabla u|^2}{A-u} + \frac{|\nabla u|^2 \overline{\nabla} u}{(A-u)^2}) + \overline{\nabla}|S|_h^{2\lambda_4} (\frac{\nabla|\nabla u|^2}{A-u} + \frac{|\nabla u|^2 \nabla u}{(A-u)^2})]\\
& -|S|_h^{2\lambda_4} (\frac{\nabla|\nabla u|^2 \cdot \overline{\nabla} u}{(A-u)^2} + \frac{\overline{\nabla}|\nabla u|^2 \cdot \nabla u}{(A-u)^2}) + 2|S|_h^{2\lambda_4} \frac{\nabla u|^4}{(A-u)^3}.
\end{aligned}$$

Also

$$\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)|S|_h^{2\lambda_4} tr_\omega(\chi)\\
= \;& |S|_h^{2\lambda_4} (\frac{\partial}{\partial t} - \Delta) tr_\omega(\chi) - \Delta|S|_h^{2\lambda_4} tr_\omega(\chi) - (\nabla|S|_h^{2\lambda_4} \cdot \overline{\nabla} tr_\omega(\chi) + \overline{\nabla}|S|_h^{2\lambda_4} \cdot \nabla tr_\omega(\chi))
\end{aligned}$$

and

$$\nabla H = (|S|_h^{2\lambda_4} \frac{\nabla|\nabla u|^2}{A-u} - |S|_h^{2\lambda_4} \frac{|\nabla u|^2 \nabla u}{(A-u)^2}) + \nabla|S|_h^{2\lambda_4} \frac{|\nabla u|^2}{A-u} + \nabla|S|_h^{2\lambda_4} tr_\omega(\chi) + |S|_h^{2\lambda_4} \nabla tr_\omega(\chi).$$

Since $tr_\omega(\chi) \leq C$ and $|S|_h^2$ can be considered as functions pulled back from the base, one can easily show that

$$|\nabla|S|_h^{2\lambda_4}|^2 \leq C|S|_h^{4\lambda_4 - 2} tr_\omega(\chi)$$

and

$$|\Delta|S|_h^{2\lambda_4}| \leq C|S|_h^{2\lambda_4 - 2} tr_\omega(\chi).$$

Also note that $|S|_h^{2\delta}u$ is a bounded function on X for any $\delta > 0$. Calculate

$$
\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)H \\
\leq\ & |S|_h^{2\lambda_4-2}(C(\epsilon)(\frac{|\nabla u|^2}{A-u} + \frac{|\nabla u|^2}{(A-u)^2}) + \epsilon|\nabla tr_\omega(\chi)|^2) + C(\epsilon)|S|_h^{2\lambda_4-1}\frac{|\nabla u|^3}{(A-u)^2}| \\
& +(\frac{\partial}{\partial t} - \Delta)|S|_h^{2\lambda_4}tr_\omega(\chi) + \epsilon|S|_h^{2\lambda_4}|\nabla tr_\omega(\chi)|^2 + C(\epsilon) \\
& +(1-\epsilon)(\nabla H \cdot \frac{\overline{\nabla} u}{A-u} + \overline{\nabla} H \cdot \frac{\nabla u}{A-u}) \\
& -\frac{1}{2}|S|_h^{2\lambda_4}\frac{(|\nabla\nabla u|^2 + |\overline{\nabla}\nabla u|^2)}{A-u} - \epsilon|S|_h^{2\lambda_4}\frac{|\nabla u|^4}{(A-u)^3}
\end{aligned}
$$

for small $\epsilon > 0$. Similar calculation as in the proof of the Schwarz lemma shows that

$$
(\frac{\partial}{\partial t} - \Delta)|S|_h^{2\lambda_4}tr_\omega(\chi) + \epsilon|S|_h^{2\lambda_4}|\nabla tr_\omega(\chi)|^2 \leq C.
$$

At any point (t_0, z_0) H achieves its maximum , by the maximum principle one has $\nabla H = 0$ and

$$
0 \leq -\frac{\epsilon}{2}|S|_h^{2\lambda_4}\frac{|\nabla u|^4}{(A-u)^3}(t_0, z_0) + C.
$$

Therefore $H(t_0, z_0) \leq C$ and

$$
H \leq C.
$$

Now we can prove the second inequality by making use of the first one. Let $K = |S|_h^{2\lambda_5}\frac{A-\Delta u}{A-u}$ with $\Delta u \leq A-1$. Then $\max K$ is uniformly bounded below from zero. By standard calculation we have

$$
\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)K \\
=\ & |S|_h^{2\lambda_5}\frac{(|\overline{\nabla}\nabla u|^2 - \Delta u + g^{i\bar{l}}g^{k\bar{j}}\chi_{i\bar{j}}u_{k\bar{l}} - \Delta tr_\omega(\chi))}{A-u} \\
& +\Delta|S|_h^{2\lambda_5}\frac{\Delta u}{A-u} + [(\overline{\nabla}|S|_h^{2\lambda_5})\cdot\overline{\nabla}(\frac{\Delta u}{A-u}) + (\overline{\nabla}|S|_h^{2\lambda_5})\cdot\nabla(\frac{\Delta u}{A-u})] \\
& +|S|_h^{2\lambda_5}\frac{\nabla(\Delta u)\cdot\overline{\nabla} u + \overline{\nabla}(\Delta u)\cdot\nabla u}{(A-u)^2} + 2|S|_h^{2\lambda_5}\frac{\Delta u|\nabla u|^2}{(A-u)^3} + (\frac{\partial}{\partial t} - \Delta)|S|_h^{2\lambda_5}\frac{A}{A-u}.
\end{aligned}
$$

and

$$
\begin{aligned}
& (\frac{\partial}{\partial t} - \Delta)(K+3H) \\
\leq\ & (\nabla(K+3H)\cdot(\frac{\overline{\nabla} u}{A-u} + \frac{\overline{\nabla}|S|_h^{2\lambda_5}}{|S|_h^{2\lambda_5}}) + \overline{\nabla}(K+3H)\cdot(\frac{\nabla u}{A-u} + \frac{\nabla|S|_h^{2\lambda_5}}{|S|_h^{2\lambda_5}})) \\
& +|S|_h^{2\lambda_5}\frac{(|\overline{\nabla}\nabla u|^2 - \Delta u + g^{i\bar{l}}g^{k\bar{j}}\chi_{i\bar{j}}u_{k\bar{l}} - \Delta tr_\omega(\chi))}{A-u} - C|S|_h^{2\lambda_5-2}\frac{\Delta u}{A-u} \\
& -2|S|_h^{2\lambda_5}\frac{|\nabla\nabla u|^2 + |\overline{\nabla}\nabla u|^2}{A-u} + C \\
\leq\ & (\nabla(K+3H)\cdot(\frac{\overline{\nabla} u}{A-u} + \frac{\overline{\nabla}|S|_h^{2\lambda_5}}{|S|_h^{2\lambda_5}}) + \overline{\nabla}(K+3H)\cdot(\frac{\nabla u}{A-u} + \frac{\nabla|S|_h^{2\lambda_5}}{|S|_h^{2\lambda_5}})) \\
& -C|S|_h^{2\lambda_5-2}\frac{\Delta u}{A-u} - \frac{3}{2}|S|_h^{2\lambda_5}\frac{|\nabla\nabla u|^2 + |\overline{\nabla}\nabla u|^2}{A-u}.
\end{aligned}
$$

Here we make use of the fact that

$$\begin{aligned}-\Delta tr_\omega(\chi) &\leq R_{i\bar{j}}\chi_{i\bar{j}} + C tr_\omega^2(\chi)\\ &= -u_{i\bar{j}}\chi_{i\bar{j}} - \chi_{i\bar{j}}^2 + C\\ &\leq C(|\overline{\nabla}\nabla u| + 1).\end{aligned}$$

At any point (t_0, z_0) where H achieves its maximum, by the maximum principle one has $\nabla(K + 3H) = 0$ and

$$|S|_h^{2\lambda_5}\frac{|\nabla\nabla u|^2 + |\overline{\nabla}\nabla u|^2}{A-u}(t_0, z_0) \leq C.$$

Hence $K(t_0, z_0) \leq C$. Since H is always bounded, one has $K(t, z) \leq C$ for any (t, z). □

By the volume estimate we have the following immediate corollary.

Corollary 4.3 *For any $\delta > 0$, there exists $C > 0$ depending on δ such that*

1. $|S|_h^{2\lambda_4+\delta}|\nabla u|^2 \leq C$,
2. $-|S|_h^{2\lambda_5+\delta}\Delta u \leq C$.

Now we are in the position to prove a uniform bound for the scalar curvature. The following corollary tells that the Kähler-Ricci will collapse with bounded scalar curvature away from the singular fibres.

Corollary 4.4 *Along the Käher-Ricci flow (1.1) the scalar curvature R is uniformly bounded on any compact set of X_{reg}. More precisely, there exist constants $\lambda_6 > 0$ and C such that*

$$-C \leq R \leq \frac{C}{|S|_h^{2\lambda_6}}. \tag{4.17}$$

Proof It suffices to give an upper bound for R by Proposition 2.1. Notice that $R_{i\bar{j}} = -u_{i\bar{j}} - \chi_{i\bar{j}}$ and then

$$R = -\Delta u - tr_\omega(\chi).$$

By Corollary 4.3 and the partial second order estimate, we have

$$R \leq \frac{C}{|S|_h^{2\lambda_6}}.$$

□

It will be interesting to know if sectional curvatures are uniformly bounded on any compact set of X_{reg}. It is not expected to be true for higher dimension. For example, we can choose $X = X_1 \times X_2$ where X_1 is a Calabi-Yau manifold and X_2 is a compact Kähler manifold of $c_1(X_2) < 0$. We can also choose the initial metric $\omega_0(x_1, x_2) = \omega_1(x_1) + \omega_2(x_2)$ where $Ric(\omega_1) = 0$ and $Ric(\omega_2) = -\omega_2$. Then along the Kähler-Ricci flow (1.1), the solution $\omega(t, \cdot)$ is given by

$$\omega(t, x_1, x_2) = e^{-t}\omega_1(x_1) + \omega_2(x_2).$$

The bisectional curvature of ω_t will blow up along time if the bisectional curvature of ω_1 on X_1 does not vanish.

4.4 The second order estimates

In this section, we prove a second order estimate for the potential φ along the Kähler-Ricci flow. First we will prove a formula which allows us to commute the $\partial\overline{\partial}$ operator and the push-forward operator for smooth functions on X. Integrating along each fibre with respect to the initial metric ω_0, we get a function on Σ:

$$\overline{\varphi} = \frac{1}{\mathrm{vol}(X_s)} \int_{X_s} \varphi\omega_0.$$

This can be considered as a push-forward of φ.

Lemma 4.7 *Let φ be a smooth function defined on X, then we have*

$$\partial\overline{\partial} \int_{X_s} \varphi\omega_0 = \int_{X_s} \partial\overline{\partial}\varphi \wedge \omega_0. \tag{4.18}$$

Proof It suffices to prove that the push forward and $\partial\overline{\partial}$ are commutative. Let $\pi : \mathcal{M} \to B$ be an analytic deformation of a complex manifold $M_0 = \pi^{-1}(0)$. Choose a sufficiently small neighborhood $\Delta \subset B$ such that $M_\Delta = \pi^{-1}(\Delta) = \cup\Delta \times U_i$ with local coordinates $(z_1^i, ..., z_n^i, t)$, where z^i is the coordinate on U_i and t on Δ. Now choose any test function ζ on B with supp$\zeta \subset \Delta$ and a partition of unity ρ_i with supp$\rho_i \subset \Delta \times U_i$. Let $\varphi_i = \rho_i\varphi$. We calculate

$$\begin{aligned}\int_{M_\Delta} \zeta\partial\overline{\partial}\varphi \wedge \omega &= \sum_i \int_{\Delta\times U_i} \partial\overline{\partial}\zeta \wedge \varphi_i\omega = \int_\Delta \partial\overline{\partial}\zeta(\sum_i \int_{U_i} \varphi_i\omega) \\ &= \int_\Delta \partial\overline{\partial}\zeta(\int_{M_t} \varphi\omega) = \int_\Delta f\partial\overline{\partial} \int_{M_t} \varphi\omega.\end{aligned}$$

On the other hand side we have

$$\begin{aligned}\int_{M_\Delta} \zeta\partial\overline{\partial}\varphi \wedge \omega &= \int_{M_\Delta} \zeta \sum_i (\partial\overline{\partial}\varphi_i \wedge \omega) = \sum_i \int_{\Delta\times U_i} \zeta\partial\overline{\partial}\varphi_i\omega \\ &= \int_\Delta \zeta(\sum_i \int_{U_i} \partial\overline{\partial}\varphi_i \wedge \omega) = \int_\Delta \zeta(\int_{M_t} \partial\overline{\partial}\varphi \wedge \omega).\end{aligned}$$

Therefore

$$\int_\Delta \zeta\partial\overline{\partial} \int_{M_t} \varphi\omega = \int_\Delta \zeta(\int_{M_t} \partial\overline{\partial}\varphi \wedge \omega)$$

for any testing function f and hence

$$\partial\overline{\partial} \int_{M_t} \varphi\omega = \int_{M_t} \partial\overline{\partial}\varphi \wedge \omega.$$

□

Lemma 4.8 *There exists a constant $C > 0$ such that*

$$(\frac{\partial}{\partial t} - \Delta) \log tr_{\omega_0}(\omega) \le C(tr_\omega(\omega_0) + 1). \tag{4.19}$$

Proof Choose a normal coordinate system for g_0 such that g is diagonalized. By straightforward calculation we have

$$(\frac{\partial}{\partial t}-\Delta)tr_{\omega_0}(\omega)\leq -tr_{\omega_0}(\omega)-g^{i\bar{i}}g^{k\bar{k}}g_{i\bar{j},k}g_{j\bar{i},\bar{k}}+Ctr_{\omega_0}(\omega)tr_{\omega}(\omega_0). \tag{4.20}$$

It can also be shown that

$$\begin{aligned}
|\bigtriangledown tr_{\omega_0}(\omega)|^2 &= \sum_{i,j,k}g^{k\bar{k}}g_{i\bar{i},k}g_{j\bar{j},k}\\
&\leq \sum_{i,j}(\sum_{k}g^{k}g^{k\bar{k}}|g_{j\bar{j},k}|^2)^{\frac{1}{2}}\\
&\leq (\sum_{i}(\sum_{k}g^{k\bar{k}}|g_{i\bar{i},k}|^2)^{\frac{1}{2}})^2\\
&\leq (\sum_{i}(g_{i\bar{i}})^{\frac{1}{2}}(\sum_{k}g^{i\bar{i}}g^{k\bar{k}}|g_{i\bar{i},k}|^2)^{\frac{1}{2}})^2\\
&\leq tr_{\omega_0}(\omega)\sum_{k,i}g^{i\bar{i}}g^{k\bar{k}}|g_{k\bar{i},i}|^2\\
&\leq tr_{\omega_0}(\omega)g^{i\bar{i}}g^{k\bar{k}}g_{i\bar{k},j}g_{k\bar{i},\bar{j}}.
\end{aligned} \tag{4.21}$$

Combined with the above inequalities, the lemma follows by calculating $(\frac{\partial}{\partial t}-\Delta)\log tr_{\omega_0}(\omega)$. $\square$

Lemma 4.9

$$\Delta(e^t(\varphi-\overline{\varphi}))\leq -tr_{\omega}(\omega_0)+\frac{1}{\mathrm{Vol}(X_s)}tr_{\omega}(\int_{X_s}\omega_0^2)+2e^t. \tag{4.22}$$

Proof Applying equation (4.18), we have

$$\begin{aligned}
\Delta(\varphi-\overline{\varphi}) &= tr_{\omega}((\omega-\omega_t)-tr_{\omega}(\frac{1}{\mathrm{Vol}(X_s)}\int_{X_s}\partial\overline{\partial}\varphi\wedge\omega_0)\\
&= 2-tr_{\omega}(\omega_t)-\frac{1}{\mathrm{Vol}(X_s)}tr_{\omega}(\int_{X_s}\omega\wedge\omega_0-\int_{X_s}\omega_t\wedge\omega_0)\\
&\leq -e^{-t}tr_{\omega}(\omega_0)+\frac{e^{-t}}{\mathrm{Vol}(X_s)}tr_{\omega}(\int_{X_s}\omega_0^2)+2.
\end{aligned}$$

$\square$

Theorem 4.2 (Second order estimates) *There exist constants λ_7, A and $C>0$ such that*

$$tr_{\omega_0}(\omega)(t,z)\leq Ce^{Ae^t(\varphi-\overline{\varphi})(t,z)-\frac{A}{|S|_h^{2\lambda_7}(t,z)}\inf_{X\times[0,T]}(|S|_h^{2\lambda_7}e^s(\varphi-\overline{\varphi}))}+C. \tag{4.23}$$

Proof Put $H=|S|_h^{2\lambda_7}(\log tr_{\omega_0}(\omega)-Ae^t(\varphi-\overline{\varphi}))$. We will apply the maximum principle on the

evolution of H. There exists a constant $C > 0$ such that

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)H \\
= \ & |S|_h^{2\lambda_7}(\frac{\partial}{\partial t} - \Delta)(\log tr_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})) - (\nabla H \cdot \frac{\overline{\nabla}|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}} + \overline{\nabla} H \cdot \frac{\nabla|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}}) \\
&+ \frac{|\nabla|S|_h^{2\lambda_7}|^2}{|S|_h^{2\lambda_7}}(\log tr_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})) - \Delta(|S|_h^{2\lambda_7})(\log tr_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})) \\
\leq \ & C|S|_h^{2\lambda_7} tr_\omega(\omega_0) - A|S|_h^{2\lambda_7} tr_\omega(\omega_0) - (\nabla H \cdot \frac{\overline{\nabla}|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}} + \overline{\nabla} H \cdot \frac{\nabla|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}}) \\
&+ |S|_h^{2\lambda_7}[-Ae^t(\varphi - \overline{\varphi}) - Ae^t \frac{\partial(\varphi - \overline{\varphi})}{\partial t} + \frac{1}{\mathrm{Vol}(X_s)} tr_\omega(\int_{X_s} \omega_0^2)] + Ce^t \\
&+ \frac{|\nabla|S|_h^{2\lambda_7}|^2}{|S|_h^{2\lambda_7}}(\log tr_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})) - \Delta(|S|_h^{2\lambda_7})(\log tr_{\omega_0}(\omega) - Ae^t(\varphi - \overline{\varphi})).
\end{aligned}$$

Notice that

$$\frac{1}{\mathrm{Vol}(X_s)} tr_\omega(\int_{X_s} \omega_0^2) \leq tr_\omega(\chi) \sup_{X_s}(\frac{\omega_0^2}{\omega_{SF} \wedge \chi}).$$

If we assume Lemma 5.4, then $e^t|S|_h^{2\lambda_7}(\varphi - \overline{\varphi})$, $|S|_h^{2\lambda_7}\frac{\partial(\varphi - \overline{\varphi})}{\partial t}$ and $|S|_h^{2\lambda_7} tr_\omega(\int_{X_s} \omega_0^2)$ are uniformly bounded if λ_7 is chosen to be sufficiently large. Also we have

$$\Delta|S|_h^{2\lambda_7} \leq C|S|_h^{2\lambda_7 - 2} tr_\omega(\chi)$$

and

$$\frac{|\nabla|S|_h^{2\lambda_7}|^2}{|S|_h^{2\lambda_7}} \leq C|S|_h^{2\lambda_7 - 2} tr_\omega(\chi)$$

for a uniform constant $C > 0$. Therefore we have

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)H \\
\leq \ & C|S|_h^{2\lambda_7} tr_\omega(\omega_0) - A|S|_h^{2\lambda_7} tr_\omega(\omega_0) - (\nabla H \cdot \frac{\overline{\nabla}|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}} + \overline{\nabla} H \cdot \frac{\nabla|S|_h^{2\lambda_7}}{|S|_h^{2\lambda_7}}) + Ce^t.
\end{aligned}$$

Assume H achieves its maximum at (t_0, z_0) on $[0, T] \times X$. Applying the maximum principle, we have $\nabla H(t_0, z_0) = 0$ and then

$$\{|S|_h^{2\lambda_7} tr_\omega(\omega_0)\}(t_0, z_0) \leq Ce^{t_0}.$$

This implies

$$\{|S|_h^{2\lambda_7} tr_{\omega_0}(\omega)\}(t_0, z_0) \leq C.$$

The theorem is proved then by comparing H at any point $(t, z) \in [0, T] \times X$ and (t_0, z_0). □

5 Generalized Kähler-Einstein metrics and the Kähler-Ricci flow

5.1 Limiting metrics on canonical models and Weil-Petersson metrics

Let X be a smooth projective manifold of $\dim X = n$. Suppose that μK_X is base point free for $\mu >> 1$ and $\nu(X) = \kappa$ with $0 < \kappa \leq n$. Then

$$|\mu K_X| : X \to X_\mu \subset \mathbf{CP}^{N_\mu}$$

is a holomorphic map for $\mu >> 1$. Fix such μ, we have a holomorphic fibration $f : X \to X_{can}$ such that $\mu K_X = f^*\mathcal{O}(1)$, where X_{can} is the canonical model of X and coincides with X_μ for μ sufficiently large. The abundance conjecture claims that there is such a holomorphic map f whenever K_X is nef. If K_X is also big, it was proved by Kawamata. If $\kappa = n$, X is a minimal model of general type with K_X big and nef., the Kähler-Ricci flow deforms any Kähler metric onto a unique singular Kähler-Einstein metric on X (see [Ts], [TiZha]). If $0 < \kappa < n$, then for a generic fibre X_s, one has $\mu K_{X_s} = f^*\mathcal{O}(1)$ for some $\mu \in \mathbf{N}$, thus μK_{X_s} is trivial and X_s is a Calabi-Yau. We can choose χ to be a multiple of the Fubini-Study metric of $\mathbf{CP}^{N_\mu}$ restricted on X_{can} such that $f^*\chi \in -2\pi c_1(X)$. Notice that $f^*\chi$ is a smooth semi-positive $(1,1)$-form on X. For simplicity, we sometimes denote it by χ. Denote by X^0_{can} the set of all smooth points s of X_{can} such that $X_s = f^{-1}(s)$ is a smooth fiber. Put $X_{reg} = f^{-1}(X^0_{can})$. Clearly, it is a smooth manifold.

Lemma 5.1 *For any Kähler class $[\omega]$ on X, there is a smooth function ψ such that $\omega_{SF} := \omega + \sqrt{-1}\partial\bar\partial\psi$ is a closed semi-flat $(1,1)$-form in the following sense: the restriction of ω_{SF} to each smooth $X_s \subset X_{reg}$ is Ricci flat.*

Proof On each smooth fiber X_s, let ω_s be the restriction of ω to X_s and ∂_V and $\overline{\partial}_V$ be the restriction of ∂ and $\overline{\partial}$ to X_s. Then by the Hodge theory, there is a unique function h_s on X_s defined by

$$\begin{cases} \partial_V\overline{\partial}_V h_s = -\partial_V\overline{\partial}_V \log\omega_s^{n-\kappa} \\ \int_{X_s} e^{h_s}\omega_s = \int_{X_s}\omega_s. \end{cases} \tag{5.1}$$

By Yau's solution to the Calabi conjecture, there is a unique ψ_s to the following Monge-Ampère equation

$$\begin{cases} \frac{(\omega_s+\sqrt{-1}\partial_V\overline{\partial}_V\psi_s)^{n-\kappa}}{\omega_s^{n-\kappa}} = e^{h_s} \\ \int_{X_s}\psi_s\omega_s^{n-\kappa} = 0. \end{cases} \tag{5.2}$$

Since f is holomorphic, $\psi(z,s) = \psi_s(z)$ is well-defined as a smooth function on X_{reg}. □

Remark 5.2 The function ψ extends continuously to each smooth fiber X_s even if s is a singular point of X_{can}.

By the Hodge theory, there exists a volume form on X such that $\sqrt{-1}\partial\overline{\partial}\log\Omega = \chi$. Define

$$F = \frac{\Omega}{\begin{pmatrix} n \\ \kappa \end{pmatrix}\omega_{SF}^{n-\kappa}\wedge\chi^\kappa}. \tag{5.3}$$

We will show properties of F and examine how F behaves near singular fibers. Define $h(z,s) = h_s(z)$, then

$$F = \frac{\Omega}{\begin{pmatrix} n \\ \kappa \end{pmatrix}\omega^{n-\kappa}\wedge\chi^\kappa}e^{-h} \geq 0. \tag{5.4}$$

It follows that F has at most poles along $X_{can}\backslash X^0_{can}$.

Lemma 5.2 *F is the pullback of a function on X_{can}. Furthermore, there exists $\epsilon > 0$ such that*

$$F \in L^{1+\epsilon}(X_{can}). \tag{5.5}$$

Proof Since χ is the pullback from X_{Can}, we have

$$\sqrt{-1}\partial_V\overline{\partial}_V \log\Omega = \sqrt{-1}\partial_V\overline{\partial}_V \log\omega_{SF}^{n-\kappa}\wedge\chi^\kappa = 0$$

on each smooth fibre X_s. Thus F is constant along each smooth fibre X_s and so it is the pullback of a function from X_{can}. Now we prove the second statement.

$$\begin{aligned}\int_{X_{can}} F^{1+\epsilon}\chi^\kappa &= \frac{1}{\int_{X_s}\omega_{SF}^{n-\kappa}}\int_X F^{1+\epsilon}\chi^\kappa\wedge\omega_{SF}^{n-\kappa}\\ &= \frac{1}{\begin{pmatrix} n\\ \kappa\end{pmatrix}\int_{X_s}\omega_{SF}^{n-\kappa}}\int_X F^\epsilon\Omega \leq C.\end{aligned}$$

The last inequality holds for sufficiently small $\epsilon > 0$ because F can have at worse pole singularities. □

There is a canonical hermitian metric on the push-forward of the dualizing sheaf $f_*(\Omega^{n-\kappa}_{X/X_{can}}) = (f_{*1}\mathcal{O}_X)^\vee$ over X^0_{can}.

Definition 5.1 *Let X be a projective manifold of complex dimension n. Suppose its canonical line bundle K_X is semi-positive and $0 < \kappa = \nu(X) < n$. Let X_{can} be the canonical model of X by pluricanonical maps. We define a canonical hermitian metric h_{can} on $f_*(\Omega^{n-\kappa}_{X/X_{can}})$ in the way that for any smooth $(n-\kappa, 0)$-form η on a smooth fiber X_s,*

$$|\eta|^2_{h_{can}} = \frac{\eta\wedge\bar\eta\wedge\chi^\kappa}{\omega_{SF}^{n-\kappa}\wedge\chi^\kappa} = \frac{\int_{X_s}\eta\wedge\bar\eta}{\int_{X_s}\omega_{SF}^{n-\kappa}}. \tag{5.6}$$

Now let us recall some facts on the Weil-Petersson metric on the moduli space $\mathcal{M}$ of polarized Calabi-Yau manifolds of dimension $n-\kappa$. Let $\mathcal{X}\to\mathcal{M}$ be a universal family of Calabi-Yau manifolds. Let $(U; t_1, ..., t_m)$ be a local holomorphic coordinate chart of $\mathcal{M}$, where $m = \dim\mathcal{M}$. Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $\iota(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through he Kodaira-Spensor map ι. The Weil-Petersson metric is defined by the L^2-inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. In the case of Calabi-Yau manifolds, we can express it as follows: Let Ψ be a nonzero holomorphic $(n-\kappa, 0)$-form on the fibre $\mathcal{X}_t$ and $\Psi\lrcorner\iota(\frac{\partial}{\partial t_i})$ be the contraction of Ψ and $\frac{\partial}{\partial t_i}$. Then the Weil-Petersson metric is given by

$$\Big(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar t_j}\Big)_{\omega_{WP}} = -\frac{\int_{\mathcal{X}_t}\Psi\lrcorner\iota(\frac{\partial}{\partial t_i})\wedge\overline{\Psi\lrcorner\iota(\frac{\partial}{\partial t_i})}}{\int_{\mathcal{X}_t}\Psi\wedge\overline{\Psi}}. \tag{5.7}$$

One can also represent ω_{WP} as the curvature form of the first Hodge bundle $f_*\Omega^{n-\kappa}_{\mathcal{X}/\mathcal{M}}$. Let Ψ be a nonzero local holomorphic section of $f_*\Omega^{n-\kappa}_{\mathcal{X}/\mathcal{M}}$ and one can define the hermitian metric h_{WP} on $f_*\Omega^{n-\kappa}_{\mathcal{X}/\mathcal{M}}$ by

$$|\Psi_t|^2_{h_{WP}} = \int_{\mathcal{X}_t}\Psi_t\wedge\overline{\Psi_t}. \tag{5.8}$$

Then the Weil-Petersson metric is given by

$$\omega_{WP} = Ric(h_{WP}). \tag{5.9}$$

Lemma 5.3

$$Ric(h_{can}) = \omega_{WP}. \tag{5.10}$$

Proof Let $u = \frac{\Psi\wedge\overline{\Psi}}{\omega_{SF}^{n-\kappa}}$. Notice Ψ restricted on each fibre $\mathcal{X}_t$ is a holomorphic $(n-\kappa, 0)$-form and $\Psi\wedge\overline{\Psi}$ is a Calabi-Yau volume form, therefore u is constant along each fibre and can be considered as the pullback of a function on $\mathcal{M}$. Then by definition

$$\omega_{WP} = -\sqrt{-1}\partial\overline{\partial}\log\int_{\mathcal{X}_t} u\omega_{SF}^{n-\kappa} = -\sqrt{-1}\partial\overline{\partial}\log u,$$

where the last equality makes use of the fact that $\int_{\mathcal{X}_t}\omega_{SF}^{n-\kappa} = constant$. At the same time

$$Ric(h_{can}) = -\sqrt{-1}\partial\overline{\partial}\log\frac{\Psi\wedge\overline{\Psi}\wedge\chi^\kappa}{\omega_{SF}^{n-\kappa}\wedge\chi^\kappa} = -\sqrt{-1}\partial\overline{\partial}\log u.$$

This proves the lemma. $\square$

Definition 5.2 (**Canonical metrics on** X_{can}) *We define the generalized Kähler-Einstein metric ω in the class of $-2\pi f_* c_1(X)$ on X_{can}^0 with respect to the fibration $f : X \to X_{can}$ by*

$$Ric(\omega) = -\omega + \omega_{WP}. \tag{5.11}$$

In general if $X \to \Sigma$ is a Calabi-Yau fibration, we can define a generalized Kähler-Einstein metric $\lambda\omega \in 2\pi c_1(\Sigma) + 2\pi c_1(f_*\Omega_{X/\Sigma}^{n-\kappa})$ by

$$Ric(\omega) = \lambda\omega + \omega_{WP}, \tag{5.12}$$

where $\lambda = -1,\ 0,\ 1$.

The following theorem is the main result of this section and its proof is essentially due to the work of Kolodziej [Kol1, Kol2].

Theorem 5.1 *Suppose that X_{can} is smooth (or has at worst orbifold singularities), then there is a unique solution $\varphi_\infty \in \mathrm{PSH}(\chi)\cap C^0(X_{can})$ of the following equation on X_{can}*

$$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^\kappa = F\, e^{\varphi}\chi^\kappa. \tag{5.13}$$

Furthermore, $\omega = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$ is a positive closed current on X_{can}. If ω is smooth on X_{can}^0, then the Ricci curvature of ω on X_{can}^0 is given by

$$Ric(\omega) = -\chi_\omega + \omega_{WP}. \tag{5.14}$$

In fact, these canonical metrics belong to a class of Kähler metrics which generalize Calabi's extremal metrics. Let Y be a Kähler manifold of complex dimension n together with a fixed closed (1,1)-form θ. Fix a Kähler class $[\omega]$, denote by $\mathcal{K}_{[\omega]}$ the space of Kähler metrics within the same Kähler class, that is, all Kähler metrics of the form $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. One may consider the following equation:

$$\bar{\partial}V_\varphi = 0, \tag{5.15}$$

where V_φ is defined by

$$\omega_\varphi(V_\varphi, \cdot) = \bar{\partial}(S(\omega_\varphi) - tr_{\omega_\varphi}(\theta)). \tag{5.16}$$

Clearly, when $\theta = 0$, (5.15) is exactly the equation for Calabi's extremal metrics. For this reason, we call a solution of (5.15) a generalized extremal metric. If Y does not admit any nontrivial holomorphic vector fields, then any generalized extremal metric ω_φ satisfies

$$S(\omega_\varphi) - tr_{\omega_\varphi}(\theta) = \mu,$$

where μ is the constant given by

$$\mu = \frac{n(2\pi c_1(Y) - [\theta]) \cdot [\omega]^{n-1}}{[\omega]^n}.$$

Moreover, if $2\pi c_1(Y) - [\theta] = \lambda[\omega]$, then any such a metric satisfies

$$Ric(\omega_\varphi) = \lambda\omega_\varphi + \theta,$$

that is, ω_φ is a generalized Kähler-Einstein metric. This can be proved by an easy application of the Hodge theory. More interestingly, if we take θ to be the pull-back of ω_{WP} by $f : X^0_{can} \to \mathcal{M}_{CY}$, then we get back those generalized Kähler-Einstein metrics which arise from limits of the Kähler-Ricci flow.

5.2 Minimal surfaces of general type

We start with minimal surfaces of general type. Let X_{can} be the canonical model of a minimal surface of general type from the contraction map $f : X \to X_{can}$. Possibly, X_{can} has rational singularities of $A-D-E$-type by contracting the (-2)-curves. Since K_{can} is ample and $f^*K_{can} = K_X$, we can assume the smooth closed $(1,1)$-form $\chi = f^*\chi \in -2\pi c_1(X)$ and χ is a Kähler from on X_{can}. It is shown in [TiZha] that the Kähler-Ricci flow (1.1) converges to the canonical metric g_{KE} on X, which is the pullback of the smooth orbifold Kähler-Einstein metric on the canonical model X_{can}, although g_{KE} might vanish along those (-2)-curves.

5.3 Minimal elliptic surfaces of Kodaira dimension one

Now consider minimal elliptic surfaces. From Lemma 5.1, we know that there exists a closed semi-flat $(1,1)$-form ω_{SF} in $[\omega_0]$.

Lemma 5.4 *Let F be the function on Σ defined by $F = \frac{\Omega}{2\omega_{SF}\wedge\chi}$ as (5.3). Let $B \subset \Sigma$ be a small disk with center 0 such that all fibres X_s, $s \neq 0$, are smooth. There exists a constant $C > 0$ such that*

1. *If X_0 is of type mI_0, then*

$$\frac{1}{C}|s|^{-\frac{2(m-1)}{m}} < F|_B \leq C|s|^{-\frac{2(m-1)}{m}}; \tag{5.17}$$

2. *If X_0 is of type mI_b or I_b^*, $b > 0$, then*

$$-\frac{1}{C}|s|^{-\frac{2(m-1)}{m}} \log|s|^2 \leq F|_B \leq -C|s|^{-\frac{2(m-1)}{m}} \log|s|^2; \tag{5.18}$$

3. *If X_0 is of any other type, then*

$$\frac{1}{C} \leq F|_B \leq C. \tag{5.19}$$

Proof Let Y be the fibration of f over B.

1. If X_0 is of type mI_0, we start with a fibration $\tilde{Y} = \mathbf{C} \times \tilde{B}/L$, where $L = \mathbf{Z} + \mathbf{Z} \cdot z(w)$ is a holomorphic family of lattices with z being a holomorphic function on $\tilde{B}$ satisfying: $z(w) = z(0) + const \cdot w^{mh}$, w is the coordinate on $\tilde{B}$, $h \in \mathbf{N}$. The automorphism of $\mathbf{C} \times B$ given by $(c, w) \to (c + \frac{1}{m}, e^{\frac{2\pi\sqrt{-1}}{m}} w)$ descends to $\tilde{Y}$ and generates a group action without fixed points. We can assume that Y is the quotient of $\tilde{Y}$ by the group action. Therefore ω_{SF} is a smooth family of Ricci-flat metrics over B. Choose a local coordinate s on B centered around 0, and a covering $\{U_\alpha\}$ of a neighborhood U of X_0 in X by small polydiscs. Since the function f^*s vanishes to order m along X_0, we can in each U_α choose a holomorphic function w_α on U_α as the mth root of f^*s, with

$$w_\alpha^m = f^*s$$

and on $U_\alpha \cap U_\beta$

$$w_\alpha = e^{\frac{2\pi\sqrt{-1}k_{\alpha\beta}}{m}} w_\beta$$

for $k_{\alpha\beta} \in \{0, 1, ..., m-1\}$. On each U_α, $ds \wedge d\overline{s} = m^2 |s|^{\frac{2(m-1)}{m}} dw_\alpha \wedge d\overline{w}_\alpha$. Then $|s|^{\frac{2(m-1)}{m}} F$ is smooth and bounded away from zero on Y. Thus (5.17) is proved.

2. If X_0 is of type I_b, $b > 0$, we can assume $Y = \mathbf{C} \times B/L$, where

$$L = \mathbf{Z} + \mathbf{Z} \frac{b}{2\pi\sqrt{-1}} \log s.$$

Let γ_0 be an arc passing through 0 in B and γ be an arc on X transverse to X_0 with $f \cdot \gamma = \gamma_0$. We also assume that γ does not pass through any double point of X_0. $\Omega = F\omega_{SF} \wedge \chi$ is smooth and non-degenerate and so is χ along γ. Since $F = \frac{\Omega}{\omega_0 \wedge \chi} \frac{\omega_0 \wedge \chi}{\omega_{SF} \wedge \chi}$, it suffices to estimate the function $\frac{\omega_0}{\omega_{SF}}|_{X_s}$ restricted to γ near X_0. Along γ, $\omega_0|_{X_s}$ pulls back to a metric uniformly equivalent to a fixed flat metric $\omega_{\mathbf{C}}$, so it suffices to estimate $\frac{\omega_{\mathbf{C}}}{\omega_{SF}}|_{X_s}$. But

$$\frac{\omega_{\mathbf{C}}}{\omega_{SF}}|_{X_s} = \frac{\int_{X_s} \omega_{\mathbf{C}}}{\int_{X_s} \omega_{SF}} = \frac{b \log |s|}{2\pi \int_{X_s} \omega_{SF}}$$

and $\mathrm{Vol}(X_s) = \int_{X_s} \omega_{SF}$ is a constant independent of s. Therefore there exists a constant C such that

$$-\frac{1}{C} \log |s|^2 \leq F \leq -C \log |s|^2.$$

If X_0 is of type mI_b, $b > 0$, we start with a fibration $f : \tilde{Y} \to \tilde{B}$, where $\tilde{Y} = \mathbf{C} \times \tilde{B}/L$ and $L = \mathbf{Z} + \mathbf{Z} \frac{mb}{2\pi\sqrt{-1}} \log w$ and w is the coordinate function of B. So $\tilde{Y_0} = C_1 + C_2 + ... + C_{mb}$ is of type I_{mb}. The automorphism $(c, w) \to (c, e^{\frac{2\pi\sqrt{-1}}{m}} w)$ of $\mathbf{C} \times \tilde{B}$ induces a fibre-preserving automorphism of order m on $\tilde{Y}$. Such an automorphism generates a group action on $\tilde{Y}$ without fixed points and the quotient of $\tilde{Y}$ has a singular fibre of type mI_b. Then by using the same arguments for singular fibres of type mI_0, we can prove (5.18). A fibration of type I_b^* $(b > 0)$ is obtained by taking a quotient of a fibration of type I_{2b} after resolving the A_1-singularities. The lattices can be locally written as $L = s^{\frac{1}{2}}\mathbf{Z} + \mathbf{Z} s^{\frac{1}{2}} \frac{b}{2\pi\sqrt{-1}} \log s$. Then the above argument gives the required estimate for F.

3. If X_0 is not of type mI_b, $b \geq 0$ or I_b^*, $b > 0$, it must be of type I_0^*, II, III, IV, IV^*, III^* or II^*. Such a singular fibre is not a stable fiber. By the table of Kodaira (cf. [Ko]), the functional invariant $J(s)$ is bounded near 0 and $J(0) = 0$ or 1. One can write down the table of local lattices of periods and the periods are bounded near the singular fibre. For example, if X_0 is of type II, then X_0 is a cuspidal rational curve with $J(s) = s^{3m+1}$, $m \in \mathbf{N} \cup \{0\}$ in the local normal representation. On each fibre X_s the above fixed flat metric $\omega_{\mathbf{C}}$ on $\mathbf{C}$ has uniformly bounded area, therefore

$$0 < \frac{1}{C} \leq \frac{\omega_{\mathbf{C}}}{\omega_{SF}}|_{X_s} = \frac{\int_{X_s} \omega_{\mathbf{C}}}{\int_{X_s} \omega_{SF}} \leq C.$$

The estimate is then proved by the same argument as that in the previous case. □

Lemma 5.5 *There is a unique solution φ_∞ of the following equation on Σ*

$$\chi + \sqrt{-1}\partial\overline{\partial}\varphi = Fe^{\varphi}\chi \tag{5.20}$$

satisfying $\sup_\Sigma |\varphi| \leq C$. *Furthermore, we have $\varphi_\infty \in C^0(\Sigma_{reg}) \cap C^\infty(\Sigma_{reg})$.*

Proof This is a corollary of Theorem 5.2, but still we give an elementary proof for the sake of completeness. Rewrite equation (5.20) as

$$\Delta\varphi = Fe^{\varphi} - 1, \tag{5.21}$$

where Δ is the Laplacian operator with respect to χ. We will apply the method of continuity to find the solutions of the following equation parameterized by $t \in [0, 1]$:

$$\Delta\varphi = e^{\varphi}(\frac{2\omega_{SF} \wedge \chi}{\Omega} + t)^{-1} - 1. \tag{5.22}$$

Obviously equation (5.22) is solvable for all $t \in (0, 1]$. To solve for $t = 0$ we need to derive the uniform C^0-estimate for φ_t. By the maximum principle,

$$\sup_{\Sigma \times (0,1]} \varphi_t \leq \sup_{\Sigma} \log \frac{2\omega_{SF} \wedge \chi}{\Omega} \leq C.$$

By Lemma 5.4, $||F||_{L^p}$ is bounded for some $p > 1$, then the standard L^p estimate gives

$$||\varphi_t||_{L^p_2} \leq C(||F||_{L^p} + 1) \leq C.$$

The Sobolev embedding theorem implies

$$||\varphi_t||_{L^\infty} \leq C$$

for $t \in (0, 1]$. With the C^0 estimate, we can derive the uniform C^k-estimate for φ_t by the local estimates of the standard theory of linear elliptic PDE due to the fact that Δ has uniformly bounded coefficients. Therefore there exists $\varphi_\infty \in C^\infty(\Sigma_{reg})$ satisfying equation (5.20).

Now we prove the uniqueness. Suppose there is another solution ψ solving (5.13) with $\sup_\Sigma |\psi| \leq C$. We define

$$\varphi_\epsilon = \varphi + \epsilon \log |S|_h^2$$

such that $|S|_h^2 \leq 1$. Then

$$\sqrt{-1}\partial\overline{\partial}(\varphi_\epsilon - \psi) = (\frac{1}{|S|_h^{2\epsilon}} e^{\varphi_\epsilon - \psi} - 1)(\chi + \sqrt{-1}\partial\overline{\partial}\psi) - \epsilon\chi.$$

Note that $\{\varphi_\infty - \psi \geq 0\} \cap (\cup_i X_{p_i}) = \phi$, then

$$\begin{aligned}
& -\int_{\{\varphi_\epsilon-\psi\geq 0\}} |\nabla_\psi(\varphi_\epsilon-\psi)|^2(\chi+\sqrt{-1}\partial\overline{\partial}\psi) \\
= & \int_{\{\varphi_\epsilon-\psi\geq 0\}} (\varphi_\epsilon-\psi)\sqrt{-1}\partial\overline{\partial}(\varphi_\epsilon-\psi) \\
= & \int_{\{\varphi_\epsilon-\psi\geq 0\}} (\varphi_\epsilon-\psi)(\frac{1}{|S|_h^{2\epsilon}}e^{\varphi_\epsilon-\psi}-1)(\chi+\sqrt{-1}\partial\overline{\partial}\psi) - \int_{\{\varphi_\epsilon-\psi\geq 0\}} \epsilon\chi \\
\geq & -C\epsilon.
\end{aligned}$$

Let $\epsilon \to 0$, by Fatou's lemma,

$$\int_{\{\varphi-\psi\geq 0\}} |\nabla_\psi(\varphi_\epsilon-\psi)|^2(\chi+\sqrt{-1}\partial\overline{\partial}\psi) = 0,$$

therefore $\varphi-\psi=0$ on each path connected component of $\{\varphi-\psi\geq 0\}$. On the other hand, we can apply the same argument for ψ_ϵ and it infers that

$$\varphi-\psi=0.$$

$\square$

Corollary 5.1 *Let $f : X \to \Sigma$ be a minimal elliptic surfac e of $\nu(X) = 1$ with singular fibres $X_{s_1} = m_1F_1$, ... , $X_{s_k} = m_kF_k$ with multiplicity $m_i \in \mathbf{N}$, $i = 1, ..., k$. If φ_∞ is the solution in Lemma 5.5, then $\omega_\infty = \chi+\sqrt{-1}\partial\overline{\partial}\varphi_\infty$ is a Kähler form on Σ and Ricci curvature of ω_∞ is given by the following formula*

$$Ric(\omega_\infty) = -\omega_\infty + \omega_{SF} + \sum_{i=1}^{k}\frac{m_k-1}{m_k}[s_i], \tag{5.23}$$

where ω_{WP} is the induced Weil-Petersson metric and $[s_i]$ is the current of integration associated to the divisor s_i on Σ. In particular, if $f : X \to \Sigma$ has only singular fibres of type mI_0, then χ_∞ is a hyperbolic cone metric on X given by

$$Ric(\chi_\infty) = -\omega_\infty. \tag{5.24}$$

This tells us that ω_∞ is canonical in the sense that it is more or less a hyperbolic metric with a correction term $\omega_{SF} + \sum_{i=1}^{k}\frac{m_k-1}{m_k}[s_i]$ inherited from the elliptic fibration structure of X. Also we notice that the residues only come from multiple fibres.

6 Convergence

6.1 Sequential convergence

In this section we will prove a sequential convergence of the Kähler-Ricci flow to a canonical metric on the base Σ.

Lemma 6.1 *There exist constants λ_8 and C such that*

$$\left|\nabla_{g_0}\frac{\partial\varphi}{\partial t}\right|^2 \leq Ce^{\frac{C}{|S|_h^{2\lambda_8}}}. \tag{6.1}$$

Proof Put $u = \frac{\partial\varphi}{\partial t} + \varphi$ and calculate

$$\begin{aligned}
& \frac{1}{2}\left|\nabla_{g_0}\frac{\partial\varphi}{\partial t}\right|^2 - |\nabla_{g_0}\varphi|^2 \\
\leq & \left|\nabla_{g_0}(\frac{\partial\varphi}{\partial t} + \varphi)\right|^2 = \frac{\sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega_0}{\omega_0^2} = |\nabla u|^2 \frac{\omega \wedge \omega_0}{\omega_0^2} \\
\leq & |\nabla u|^2 \, tr_{\omega_0}(\omega).
\end{aligned}$$

The lemma follows from Theorem 4.2(the second order estimate), Corollary 4.2 and Theorem 4.1 (the gradient estimate) since $|\nabla_{g_0}\varphi|$ can be bounded by $tr_{\omega_0}(\omega)$. $\square$

We first prove a weak convergence for $\frac{\partial\varphi}{\partial t}$ by picking a sequence.

Lemma 6.2 *There exists a sequence t_j such that $\frac{\partial\varphi}{\partial t}(t_j, \cdot)$ converges to 0 weakly. Thus by Lemma 6.1, $\frac{\partial\varphi}{\partial t}(t_j, \cdot)$ converges to 0 in $C^{0,\alpha}$ for any $0 < \alpha < 1$ on any compact set of X_{reg}.*

Proof For any $T_2 > T_1 > 0$ and $z \in X$ we have

$$\int_{T_1}^{T_2} \frac{\partial\varphi}{\partial t}(s, z) ds = \varphi(T_2, z) - \varphi(T_1, z).$$

Since $|\varphi|_{C^0}$ is uniformly bounded on X, we have

$$\left|\int_{T_1}^{T_2} \frac{\partial\varphi}{\partial t}(s, z) ds\right| \leq C.$$

Choose a countable basis $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ for the topology of X. Then on each U_α, $\int_{T_1}^{T_2}(\int_{U_\alpha} \frac{\partial\varphi}{\partial t}\omega_0^2)ds$ is uniformly bounded independent of the choice of T_1 and T_2. Therefore by applying the mean value theorem with $T_1, T_2, T_2 - T_1 \to \infty$, we can show that by passing to a sequence $\frac{\partial\varphi}{\partial t}(t_{k,\alpha}, \cdot)$

$$\lim_{k\to\infty} \int_{U_\alpha} \frac{\partial\varphi}{\partial t}(t_{k,\alpha}, \cdot) = 0.$$

By taking the diagonal sequence t_j of $t_{k,\alpha}$, one has for each $\alpha \in \mathcal{A}$

$$\lim_{j\to\infty} \int_{U_\alpha} \frac{\partial\varphi}{\partial t}(t_j, \cdot)\omega_0^2 = 0.$$

Therefore $\frac{\partial\varphi}{\partial t}(t_j, \cdot)$ converges weakly to 0 on X. $\square$

Theorem 6.1 *There exists a sequence $\{t_j\}_{j=1}^\infty$ with $t_j \to \infty$ such that $\varphi(t_j, \cdot)$ converges to the pullback of a function φ_∞ given by equation (5.20) uniformly on any compact set of X_{reg} in the sense of $C^{1,1}$.*

Proof Consider any test function $\zeta \in C_0^\infty(\Sigma_{reg})$. We calculate

$$\begin{aligned}
\int_X \zeta e^t \omega^2 = & \int_X \zeta \left[2(\chi + \sqrt{-1}\partial\overline{\partial}\bar{\varphi}) \wedge \omega_0 + 2\sqrt{-1}e^t\partial\overline{\partial}(\varphi - \bar{\varphi}) \wedge (\chi + \sqrt{-1}\partial\overline{\partial}\bar{\varphi})\right. \\
& \left. + e^{-t}(\omega_0 - \chi + \sqrt{-1}e^t\partial\overline{\partial}(\varphi - \bar{\varphi}))^2\right].
\end{aligned}$$

Notice that $\int_X \zeta\partial\overline{\partial}(\varphi-\bar\varphi)\wedge(\chi+\sqrt{-1}\partial\overline{\partial}\bar\varphi)=\int_X(\varphi-\bar\varphi)\partial\overline{\partial}\zeta\wedge(\chi+\sqrt{-1}\partial\overline{\partial}\bar\varphi)=0$ and

$$\begin{aligned}
& \int_X \zeta e^{-t}(\omega_0-\chi+\sqrt{-1}e^t\partial\overline{\partial}(\varphi-\bar\varphi))^2 \\
= & \int_X \zeta\left[e^{-t}(\omega_0-\chi)^2+2\sqrt{-1}\partial\overline{\partial}(\varphi-\bar\varphi)\wedge(\omega_0-\chi)+\sqrt{-1}e^t\partial\overline{\partial}(\varphi-\bar\varphi)\wedge\sqrt{-1}\partial\overline{\partial}(\varphi-\bar\varphi)\right] \\
= & \int_X \left[2(\varphi-\bar\varphi)\sqrt{-1}\partial\overline{\partial}\zeta\wedge(\omega_0-\chi)+e^t(\varphi-\bar\varphi)\partial\overline{\partial}\zeta\wedge\sqrt{-1}\partial\overline{\partial}(\varphi-\bar\varphi)\right]+O(e^{-t}) \\
= & O(e^{-t}).
\end{aligned}$$

Therefore

$$\int_X 2\zeta(\chi+\sqrt{-1}\partial\overline{\partial}\bar\varphi)\wedge\omega_0=\int_X\zeta e^t\omega^2+O(e^{-t})=\int_X\zeta e^{\varphi+\frac{\partial\varphi}{\partial t}}\Omega+O(e^{-t}).$$

Since $\chi+\sqrt{-1}\partial\overline{\partial}\bar\varphi$ sits on the base Σ and the volume along each smooth fibre given by ω_{SF} is the same as that of ω_0, we have

$$\int_X 2\zeta(\chi+\sqrt{-1}\partial\overline{\partial}\bar\varphi)\wedge\omega_{SF}=\int_X\zeta e^{\varphi+\frac{\partial\varphi}{\partial t}}\frac{\Omega}{\chi\wedge\omega_{SF}}\chi\wedge\omega_{SF}+O(e^{-t}).$$

So for all $\zeta\in C_0^\infty(\Sigma_{reg})$, we have

$$\lim_{t_j\to\infty}\int_X 2\zeta(\chi+\sqrt{-1}\partial\overline{\partial}\bar\varphi)\wedge\omega_{SF}=\int_X\zeta e^{\varphi_\infty}\frac{\Omega}{\chi\wedge\omega_{SF}}\chi\wedge\omega_{SF}.$$

since $\frac{\partial\varphi}{\partial t}(t_j,\cdot)\to 0$. By taking the convergent subsequent we have

$$\frac{\chi+\sqrt{-1}\partial\overline{\partial}\varphi_\infty}{\chi}=e^{\varphi_\infty}\frac{\Omega}{2\chi\wedge\omega_{SF}}.$$

Notice φ_∞ is uniformly C^0 bounded since φ is uniformly bounded in $C^0(X)$ along the Kähler-Ricci flow. Also by the uniqueness of the solution for the equation above, we have proved the theorem. □

6.2 Uniform convergence

In this section we will prove a uniform convergence of the Kähler-Ricci flow. Since φ and φ_∞ are both uniformly bounded on X. Therefore for any $\epsilon>$, there exists $r_\epsilon>0$ with $\lim_{\epsilon\to 0}r_\epsilon=0$, such that for any $z\in\cup_{i=1}^{\mu}B_{r_\epsilon}(p_i)$ and $t>0$ we have

$$\begin{aligned}
&(\varphi-\varphi_\infty+\epsilon\log|S|_h^2)(t,z)<-1 \quad\text{and}\\
&(\varphi-\varphi_\infty-\epsilon\log|S|_h^2)(t,z)>1,
\end{aligned}$$

where $B_{r_\epsilon}(p_i)$ is a geodesic tube centered at the singular fibre X_{p_i} with radius r_ϵ with respect to the metric ω_0. Suppose the semi-flat closed form is given by $\omega_{SF}=\omega_0+\sqrt{-1}\partial\overline{\partial}\rho_{SF}$ and ρ_{SF} blows up near the singular fibres. We can always find an approximation ρ_ϵ for ρ_{SF} such that ρ_ϵ is smooth on X and on $X\setminus\cup_{i=0}^{\mu}B_{r_\epsilon}(p_i)$

$$\rho_\epsilon=\rho_{SF}.$$

We also define $\omega_{SF,\epsilon}=\omega_0+\sqrt{-1}\partial\overline{\partial}\rho_\epsilon$. Now we define the twisted difference of φ and φ_∞ by

$$\psi_\epsilon^-=\varphi-\varphi_\infty-e^{-t}\rho_\epsilon+\epsilon\log|S|_h^2 \quad\text{and} \tag{6.2}$$
$$\psi_\epsilon^+=\varphi-\varphi_\infty-e^{-t}\rho_\epsilon-\epsilon\log|S|_h^2. \tag{6.3}$$

Lemma 6.3 *There exists $\epsilon_0 >$ such that for any $0 < \epsilon < \epsilon_0$, there exists $T_\epsilon > 0$ such that for and $z \in X$ and $t > T_\epsilon$ we have*

$$\psi_\epsilon^-(t,z) \le 4\epsilon \quad \text{and} \tag{6.4}$$

$$\psi_\epsilon^+(t,z) \ge -4\epsilon. \tag{6.5}$$

Proof The evolution for ψ_ϵ^- is given by

$$\frac{\partial \psi_\epsilon^-}{\partial t} = \log \frac{e^t(\chi_\infty + \epsilon\chi + e^{-t}\omega_{SF,\epsilon} + \sqrt{-1}\partial\overline{\partial}\psi_\epsilon^-)^2}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2. \tag{6.6}$$

Since ρ_ϵ is bounded on X, we can always choose $T_1 > 0$ sufficiently large such that for $t > T_1$ we have $\psi_\epsilon^-(t,z) < -\frac{1}{2}$ on $\cup_{i=1}^\mu B_{r_\epsilon}(p_i)$ and $e^{-t}\frac{\omega_{SF}^2}{\chi_\infty \wedge \omega_{SF}} \le \epsilon$ on $X \setminus \cup_{i=1}^\mu B_{r_\epsilon}(p_i)$. We will discuss in two cases for $t > T_1$.

1. If $\psi_{\epsilon,max}^-(t) = \max_X \psi_\epsilon^-(t,\cdot) = \psi_\epsilon^-(t, z_{max,t}) > 0$ for all $t > T_1$. Then $z_{max,t} \in X \setminus \cup_{i=1}^\mu B_{r_\epsilon}(p_i)$ for all $t > T_1$. Applying the maximum principle at $z_{max,t}$ we have

$$\begin{aligned}\frac{\partial \psi_\epsilon^-}{\partial t}(t, z_{max,t}) &\le \{\log \frac{e^t(\chi_\infty + \epsilon\chi + e^{-t}\omega_{SF,\epsilon})^2}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2\}(t, z_{max,t}) \\ &= \{\log \frac{(\chi_\infty + \epsilon\chi) \wedge \omega_{SF,\epsilon} + e^{-t}\omega_{SF,\epsilon}^2}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2\}(t, z_{max,t}) \\ &= \{\log \frac{(\chi_\infty + \epsilon\chi) \wedge \omega_{SF} + e^{-t}\omega_{SF}^2}{\chi_\infty \wedge \omega_{SF}} - \psi_\epsilon^- + \epsilon \log |S|_h^2\}(t, z_{max,t}) \\ &\le -\psi_\epsilon^-(t, z_{max,t}) + \log(1+2\epsilon) + \epsilon.\end{aligned}$$

   Applying the maximum principle again, we have

$$\psi_\epsilon^- \le 4\epsilon + O(e^{-t}). \tag{6.7}$$

2. If there exists $t_0 \ge T_1$ such that $\max_{z\in X} \psi_\epsilon^-(t_0, z) = \psi_\epsilon^-(t_0, z_0) < 0$ for some $z_0 \in X$. Assume t_1 is the first time when $\max_{z\in X, t\le t_1} \psi_\epsilon^-(t,z) = \psi_\epsilon^-(t_1, z_1) > 4\epsilon$. Then $z_1 \in X \setminus \cup_{i=1}^\mu B_{r_\epsilon}(p_i)$ and applying the maximum principle we have

$$\begin{aligned}\psi_\epsilon^-(t_1, z_1) &\le \{\log \frac{(\chi_\infty + \epsilon\chi) \wedge \omega_{SF} + e^{-t_1}\omega_{SF}^2}{\chi_\infty \wedge \omega_{SF}} + \epsilon \log |S|_h^2\}(t_1, z_1) \\ &\le \log(1+2\epsilon) + \epsilon < 4\epsilon.\end{aligned}$$

   which contradicts the assumption that $\psi_\epsilon^-(t_1, z_1) \ge 4\epsilon$. Hence we have

$$\psi_\epsilon^- \le 4\epsilon. \tag{6.8}$$

   By the same argument we have

$$\psi_\epsilon^+ \ge -4\epsilon. \tag{6.9}$$

   This completes the proof. □

Lemma 6.4 *We have the point-wise convergence of φ on X_{reg}. Namely, for any $z \in X_{reg}$ we have*

$$\lim_{t\to\infty} \varphi(t,z) = \varphi_\infty(z). \tag{6.10}$$

Proof By lemma 6.3, we have for $t > T_\epsilon$

$$\varphi_\infty(t,z) + \epsilon \log |S|_h^2(t,z) - 4\epsilon \le \varphi(t,z) \le \varphi_\infty(t,z) - \epsilon \log |S|_h^2(t,z) + 4\epsilon. \tag{6.11}$$

Then the lemma is proved by letting $\epsilon \to 0$.

Since we have the uniform zeroth and second order estimates for φ away from the singular fibres, we derive our main theorem.

Theorem 6.2 *φ converges to the pullback of a function φ_∞ given by equation (5.13) on Σ uniformly on any compact set of X_{reg} in the sense of $C^{1,1}$.*

7 An alternative deformation and large complex structure limits

Mirror symmetry and the SYZ conjecture make predictions for Calabi-Yau manifolds with "large complex structure limit point" (cf. [StYaZa]). It is believed that in the large complex structure limit, the Ricci-flat metrics should converge in the Gromov-Hausdorff sense to a half-dimensional sphere by collapsing a special Lagrangian torus fibration over this sphere. This holds trivially for elliptic curves and is proved by Gross and Wilson (cf. [GrWi]) in the case of $K3$ surfaces. The method of the proof is to find a good approximation for the Ricci-flat metrics near the large complex structure limit. The approximation metric is obtained by gluing together the Oogrui-Vafa metrics near the singular fibres and a semi-flat metric on the regular part of the fibration. Such a limit metric of $K3$ surfaces is McLean's metric.

In this section, we will apply a deformation for a family of Calabi-Yau metrics and derive Mclean's metric [Mc] without writing down an accurate approximation metric. Such a deformation can be also done in higher dimensions. It will be interesting to have a flow which achieves this limit. The large complex structure limit of a $K3$ surface $\hat{X}$ can be identified as the mirror to the large Kähler limit of X as shown in [GrWi], so we can fix the complex structure on X and deform the Kähler class to infinity. Let $f : X \to \mathbf{CP}^1$ be an elliptic $K3$ surface. Let $\chi \ge 0$ be the pullback of a Kähler form on $\mathbf{CP}^1$ and ω_0 be a Kähler form on X. We construct a reference Kähler metric $\omega_t = \chi + t\omega_1$ and $[\omega_t]$ tends to $[\chi]$ as $t \to 0$. We can always scale ω_1 so that the volume of each fibre of f with respect to ω_t is t. Suppose Ω is a Ricci-flat volume form on X with $\partial\overline{\partial}\log\Omega = 0$. Then Yau's proof [Ya2] of Calabi's conjecture yields a unique solution φ_t to the following Monge-Ampère equation for $t \in (0,1]$

$$\begin{cases} \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^2}{\Omega} = C_t \\ \int_X \varphi_t \Omega = 0, \end{cases} \tag{7.1}$$

where $C_t = [\omega_t]^2$. Therefore we obtain a family of Ricci-flat metrics $\omega(t,\cdot) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$. The following theorem is the main result of this section.

Theorem 7.1 *Let $f : X \to \mathbf{CP}^1$ be an elliptically fibred $K3$ surface with 24 singular fibres of type I_1. Then the Ricci-flat metrics $\omega(t,\cdot)$ converges to the pullback of a Kähler metric $\tilde{\omega}$ on $\mathbf{CP}^1$ in any compact set of X_{reg} in $C^{1,1}$ as $t \to 0$. The Kähler metric $\tilde{\omega}$ on $\mathbf{CP}^1$ satisfies the equation*

$$Ric(\tilde{\omega}) = \omega_{WP}. \tag{7.2}$$

Proof All the estimates can be obtained by the same argument in Section 4 with little modification. It is relatively easy compared to the Kähler-Ricci flow because there is no such a term

of $\frac{\partial\varphi}{\partial t}$. We apply similar argument Section 6.1 to prove the weak convergence. It is not difficult to show that for any test function $\zeta \in C_0^\infty(\mathbf{CP}^1_{reg})$ we have

$$\int_X 2\zeta(\chi + \sqrt{-1}\partial\overline{\partial}\varphi_t) \wedge \omega_{SF} = \int_X \zeta\Omega + O(t).$$

By taking the convergent subsequent we have

$$\frac{\chi + \sqrt{-1}\partial\overline{\partial}\varphi_0}{\chi} = \frac{\Omega}{2\chi \wedge \omega_{SF}}.$$

Since such φ_0 with bounded $\|\varphi_0\|_{L^\infty}$ is unique and we have the 2nd order estimates for φ_t, the convergence is then uniform and this completes our proof. □

This limit metric $\tilde{\omega}$ coincides with McLean's metric as obtained by Gross and Wilson [GrWi]. Their construction is certainly much more delicate and gives an accurate approximation near the singular fibres by the Ooguri-Vafa metrics. Also Mclean's metric is an example of the generalized Kähler-Einstein metric defined in Definition 5.2 satisfying

$$Ric(\omega) = -\lambda\omega + \omega_{WP}$$

when $\lambda = 0$.

8 Generalizations and problems

8.1 A metric classification for surfaces of non-negative Kodaira dimension

In this section we will give a metric classification for surfaces of non-negative Kodaira dimension. Any surface X with $K_X \geq 0$ must be minimal and $\nu(X) > 0$.

1. When $\nu(X) = 2$, X is a minimal surface of general type and we have the following theorem.

   Theorem 8.1 [TiZha] *If X is a minimal complex surface of general type, then the global solution of the Kähler-Ricci flow converges to a positive current ω_∞ which descends to the Kähler-Einstein orbifold metric on its canonical model. In particular, ω_∞ is smooth outside finitely many rational curves and has local continuous potential.*

2. When $\nu(X) = 1$, X is a minimal elliptic surface. Theorem 1.1 shows that the Kähler-Ricci flow deforms any Kähler metric to a unique generalized Kähler-Einstein metric ω on its canonical model X_{can}.

3. When $\nu(X) = 0$, X is a Calabi-Yau surface. The Kähler-Ricci defined in [Ca] deforms any Kähler metric to a Calabi-Yau metric in the same Kähler class.

When X is not minimal, the Kähler-Ricci flow (1.1) will develop singularities at finite time. Let ω_0 be the initial Kähler metric and T be the first time such that $e^{-t}[\omega_0] - (1 - e^{-t})2\pi c_1(X)$ fails to be a Kähler class. The Kähler-Ricci flow has a smooth solution $\omega(t, \cdot)$ on $[0, T)$ converging to a degenerate metric as t tends to T (cf. [TiZha], also see [CaLa]). This degenerate metric is actually smooth outside a subvariety C. Such a C is characterized by the condition that $e^{-T}[\omega_0] - (1 - e^{-T})2\pi c_1(X)$ vanishes along C. This implies that C is a disjoint union of finitely many rational curves with self-intersection -1. Then we can blow down these (-1)-curves and

obtain a complex surface X' and $e^{-T}[\omega_0]-(1-e^{-T})2\pi c_1(X)$ descends to a Kähler class on X'. Choose a Kähler metric ω_T representing this class and then $(1-e^{-T})^{-1}(\omega_T-e^{-T}\omega_0)$ represents $-2\pi c_1(X)$. Define

$$\omega_t(\cdot)=\frac{e^{-t}-e^{-T}}{1-e^{-T}}\omega_0+\frac{1-e^{-t}}{1-e^{-T}}\omega_T$$

and write $\omega(t,\cdot)=\omega_t(\cdot)+\sqrt{-1}\partial\overline{\partial}\varphi(t,\cdot)$, then φ solves (2.8) on $[0,T)$, moreover, φ converges to a bounded function $\varphi(T,\cdot)$ as t tends to T and $\varphi(T)$ is smooth outside C. Applying the above vanishing property of C (cf. [TiZha]), one can show that $\varphi(T,\cdot)$ descends to a continuous function on X'. We believe that $\varphi(T,\cdot)$ is actually $C^{1,1}$. Since ω_T is a smooth metric on X', we can consider the flow (2.8) on X' with $\varphi(T,\cdot)$ as the initial potential. We expect that (2.8) still has a unique solution on $(0,T')$ where T' is the first time such that $e^{-t}[\omega_T]-(1-e^{-t})2\pi c_1(X')$ fails to be a Kähler class, hence, (1.1) has a smooth solution $\omega(t,\cdot)$ on $X'\times(0,T')$. Either T' is ∞ or we can repeat the previous procedure and continue the flow (1.1). After finitely many times, we will get a minimal complex surface with non-negative Kodaira dimension. Then the flow has a global solution which falls into on the cases described above. In order to complete this, one needs to do the following steps: 1. prove an optimal estimate for $\varphi(T,\cdot)$; 2. prove that the flow (2.8) has a unique solution under weaker assumptions on smoothness of initial data $\varphi(T,\cdot)$. Of course, if one can get sufficient regularity in step 1, step 2 follows from the standard theory of parabolic equations. We will address these problems in a forthcoming paper.

8.2 Higher dimension

In this section, we discuss possible generalizations of Theorem 1.1 for higher dimension. First, as we assumed in Section 5.1, let X be a non-singular variety of $\dim X=n$ such that μK_X is base point free for μ sufficiently large. Then the pluricanonical map defines a holomorphic fibration $f: X\to X_{can}$ by the linear system $|\mu K_X|$, where X_{can} is the canonical model of X.

1. If $\nu(X)=n$, K_X is big and nef. Hence X is a minimal variety of general type. The Kähler-Ricci flow will deform any Kähler metric to a canonical Kähler-Einstein metric on X (cf. [TiZha, Ts]).
2. If $\nu(X)=1$, X_{can} is a curve. With little modification of the proof, Theorem 1.1 can be generalized and the Kähler-Ricci flow will converge.
3. If $1<\mu(X)<n$, the fibration structure of f can be very complicated. A large number of the calculations can be carried out as in this paper and we expect the Kähler-Ricci flow will converge appropriately to the pullback of a canonical metric ω_∞ on the X_{can} such that $Ric(\omega_\infty)=-\omega_\infty+\omega_{WP}$ on X^0_{can}.

When K_X is nef, the Kähler-Ricci flow has long time existence, yet it does not necessarily converge, although the abundance conjecture predicts that μK_X is globally generated for μ sufficiently large. Hence, the problem of convergence of the K̈hler-Ricci flow for nef K_X can be considered as the analytic version of the abundance conjecture. If K_X is not nef, the flow will develop finite time singularities. Let ω_0 be the initial Kähler metric and T the first time such that $e^{-t}[\omega_0]-(1-e^{-t})2\pi c_1(X)$ fails to be a Kähler class. The potential $\varphi(T,\cdot)$ is bounded and smooth outside an analytic set of X(cf. [TiZha]). Let X_1 be the metric completion of $\omega(T,\cdot)$. We conjecture that this is an analytic variety. It might be a flip of X, or a variety obtained by certain standard algebraic procedure. Of course X_1 might have singularities and it is not clear at all how to develop the notion of a weak Ricci flow on a singular variety. Suppose such

a procedure can be achieved and the Kähler-Ricci flow can continue on X_1, then after applying the above procedure finitely many times on X_1, X_2, ..., X_N, we might have $K_{X_N} \geq 0$ and get the minimal model of X.

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